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PRACTICAL MECHANICS
AND
STRENGTH OF MATERIALS

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AND
STRENGTH OF MATERIALS

BY
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THIRD EDITION
TENTH IMPRESSION

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PREFACE TO THE THIRD EDITION

In presenting this third edition attention is called to the large amount of material that has been added in order to enlarge the scope and the effectiveness of the text. Many illustrative examples have been introduced, and all the lists of problems are new.

The material is arranged in a logical order and will fill the needs for a basic course. Although some of the properties of the structural sections are given, the authors believe that it is desirable for the instructor and the students to be conversant with the steel handbooks and therefore have referred to them. A list of answers to selected problems is given.

JOHN F. MANGOLD.

ILLINOIS INSTITUTE OF TECHNOLOGY,
CHICAGO, ILL.,
June, 1940.

PREFACE TO THE SECOND EDITION

This volume, as the title suggests, presents those principles of Mechanics and Strength of Materials that are believed to be essential for the practical man. The writers found a great need for such a book, while giving courses in the night school of the Armour Institute of Technology to groups of men from offices, foundries, shops and construction companies.

The aim throughout the book has been to develop Mechanics and Strength of Materials as one continuous subject, rather than two separate ones as ordinarily taught. Each chapter on Strength of Materials is preceded by those principles of Mechanics that are necessary for its presentation. For example, the chapters on Tension, Compression, Shear, and Riveted Joints follow the Mechanics of Forces in the Same Straight Line. Beams, Columns, and Torsion follow the chapters on Concurrent Forces, Moments, Center of Gravity, and Moment of Inertia.

The book is intended for night schools, vocational schools, colleges desiring a short course in Mechanics, and for a place in the library of men engaged in practical construction work. It is not a course in Design. So far as the mathematical requirements are concerned, the student who has a working knowledge of algebra, geometry and trigonometry will have no trouble in reading this volume.

Formulas for estimating the strength of parts of machines or structures are given, and are illustrated by numerous problems. The graphic and algebraic methods for the solution of problems have been presented at the same time. The constants used, taken from the tables, are only average values. The writers believe that the methods for solving the problems are so clearly presented that the reader will be able in similar cases to apply the constants that local conditions require.

The authors acknowledge their obligations to Prof. C. E. Paul for valuable suggestions and criticisms.

CHARLES W. LEIGH,
JOHN F. MANGOLD.

CHICAGO, ILL.,
January, 1930.

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PRACTICAL MECHANICS
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CHAPTER I

THE THREE SIMPLE STRESSES

1. Introduction.—The science of mechanics deals with the action of bodies on each other as well as with the action of forces on bodies. The study and analysis of this action are important to the student of engineering. Four closely interrelated factors are involved: time, space, matter, and force.

A casual consideration of the facts leads to the evident conclusion that bodies are either at rest or in motion, which at once suggests a division of mechanics into two parts: statics, which relates to bodies at rest; and kinetics, which treats of bodies in motion. The present study is concerned with statics.

In mechanics, the term *strength of materials* refers to the action on each other of the individual parts of a structure or machine when forces are applied. It also determines the dimensions of the various parts to ensure sufficient strength.

2. Force, Action, Mass.—*Force* is a term applied to any action on a body at rest that tends to make it move, or, if the body is in motion, that tends to change its velocity or its original motion or its size and shape. A force is usually thought of as a push or a pull. In the case of the attraction of a magnet for a piece of iron, the force is applied at a distance. Or it may be applied by contact, as by a hand pushing against the wall or by the pull of a rope fastened to a body. Further illustrations will readily suggest themselves, such as the wind pressure on the side of a building or against a signboard, water pressure against the face of a dam, earth pressure against a retaining wall, or steam pressure in a boiler.

The quantity of matter that a body contains is called its *mass*. Mass is constant. By means of a lever balance the mass of a body is determined by a comparison of it with a standard mass called the *pound*. In statics, the effect due to the mass of the body is entirely due to its weight.

3. Unit Measure of Force.—The unit measure of force, called a *pound force*, is the pull exerted by the earth on a pound mass. Or the pull of the earth on a mass of 1 g. may be taken as the unit; this is called a *force of* 1 *g*. Forces are measured by comparing them with one of the standard units of measure. They are expressed as pounds or grams. A force of 10 lb. means a force ten times as great as the earth's pull on a mass of 1 lb.

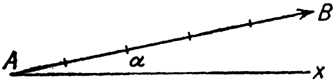


Fig. 1.

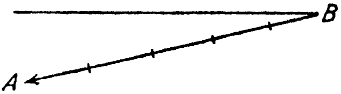


Fig. 1*a*.

4. Properties of a Force.—The effect that a force produces on a body depends upon the direction in which it acts as well as on its point of application. To be definitely known, any force must be given in terms of (*a*) magnitude in pounds, grams, or other units; (*b*) direction; (*c*) point of application on the body.

5. Vectors and Scalars.—Any quantity that has magnitude only is a *scalar; e.g.*, $10, 5 ft., etc.

Any quantity that has magnitude and direction is a *vector*. A vector is represented by a straight line drawn in the given direction and with a magnitude equal to the number of units, to any scale, contained in the quantity that it represents.

A force is therefore a *vector* and is represented by a line AB (Fig. 1). A is the point of application. The line is 5 units long and makes an angle α with a reference line AX.

Now the unit of length may represent any number of pounds. If it is 20 lb., then AB stands for $5 \times 20 = 100$ lb. If the unit is 100 lb., $AB = 5 \times 100 = 500$ lb.

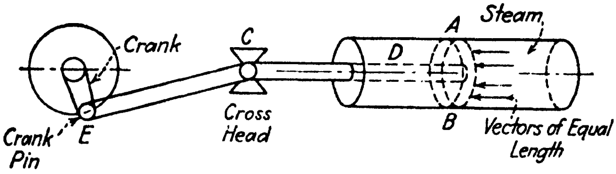


FIG. 2.

The arrowhead on the vector shows the direction in which the force acts. It is always read from the point of application toward the arrowhead. If the force in Fig. 1 were read BA, it would represent a force with B as the point of application, of magnitude 5 units and direction just the opposite to AB, as in Fig. 1a.

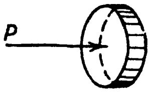


FIG. 2a.

6. External and Internal Forces.—When a force is applied to a body, it is called an *external force*. The resistance to change of motion or shape exerted by the fibers of the body is called *internal force*. In Fig. 2, when steam is admitted to the cylinder, it exerts a force on the piston AB that is an external force.

The motion of the piston and piston rod is resisted by the connecting rod EC. The connecting rod EC, therefore exerts a force at the crosshead C external to the piston and piston rod. The external forces tend to crush the piston and rod. The force exerted by the fibers of the piston and rod to keep from being crushed is an internal force. Since steam is admitted alternately, first at one end and then at the other, when the piston is moving to the right the piston rod is acted on by forces that would pull it apart. In turn the fibers of the piston exert an internal force to keep from being pulled apart. Many more examples might be given, but this one should be sufficient to give the student an idea of the meaning of the terms internal and external.

7. Concentrated and Distributed Forces.—When a force acts over a very small area, it is said to act at a point and is called a *concentrated force.*

When forces act over an area such as the piston *AB*, they are called *distributed forces.* The assumption is made that the pressure of the steam is of equal magnitude at all points of the area. The forces are then said to be *uniformly distributed.* Such forces are represented by vectors of equal length over the entire area (Fig. 2).

The pressure of the air on the water in a vessel is a uniformly distributed force, as in Fig. 3. Another familiar example is the pressure of the weight of a floor on the floor beams, as in Fig. 4.

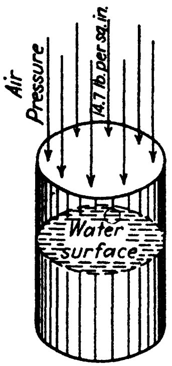


FIG. 3.

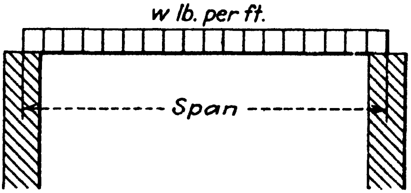


FIG. 4.

Since the area of the piston rod *CD* is small compared with the area of the piston, the pressure of the rod on the piston at *D* is considered a concentrated force and may be represented by a single vector at the center of the rod *D*, as shown in Fig. 2*a*.

8. Resultant of Two Forces in the Same Straight Line.—When two forces act in the same straight line on a body, the single force that would produce the same effect on the body as the two forces is called their *resultant.* Forces acting along the same straight line are usually called *collinear forces.*

If a body A (Fig. 5) is acted upon by forces of 20 lb. and 30 lb. as shown, it is evident that a single force of 20 lb. + 30 lb. = 50 lb. would be the resultant.

If the forces act in opposite directions (Fig. 5a), their resultant would be 30 lb. − 20 lb. = 10 lb. acting to the right.

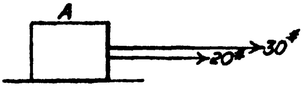


Fig. 5.

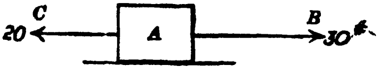


Fig. 5a.

As a matter of convenience, when a force acts in a certain direction, say to the right, it is considered *positive*. If acting to the left, it is *negative*. The force is read or expressed as so many pounds, with a plus or minus sign attached, depending on the direction in which the force acts. In Fig. 5a, $AB = +30$ lb. and $AC = -20$ lb. The function of the algebraic sign is to indicate direction of action only.

The rule for determining the resultant may be stated as follows: *The resultant of any number of forces acting in the same straight line is their algebraic sum.*

Although the choice of signs is a matter of convenience, in mechanics, the custom is to observe the signs as in mathematics. A force whose direction is from left to right is positive; a force whose direction is from right to left is negative.

Example 1.—Find the resultant of the forces in Fig. 6.

Solution.—$R = +15 + 25 - 20 - 10 = +10$. *Ans.*

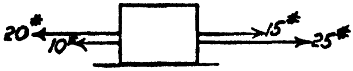


Fig. 6.

Then the four given forces may be replaced by a single force of 10 lb. acting to the right.

Example 2.—Find the resultant of the forces in Fig. 7.

Solution.—$R = +8 + 12 - 10 = +10$. *Ans.*

Then the original three forces may be replaced by a single force of 10 lb. acting vertically upward.

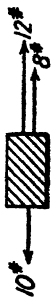


Fig. 7.

Example 3.—Find the resultant of the forces in Fig. 8.

Solution.—There are two sets of forces which will be analyzed separately.

In a vertical direction,

$$R = +10 - 10 = 0$$

In a horizontal direction,

$$R = -6 - 4 + 8 = -2 \textit{ Ans.}$$

Then the five forces may be replaced by a single force of 2 lb. acting horizontally to the left.

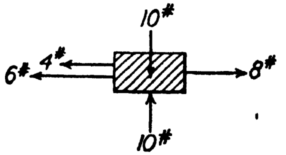


Fig. 8.

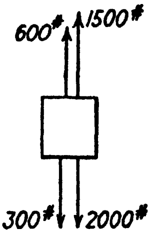


Fig. 9.

Example 4.—Find the resultant of the forces shown in Fig. 9.

Solution.—$R = +600 + 1{,}500 - 300 - 2{,}000 = -200$ lb. *Ans.* Then the original four forces may be replaced by a single force of 200 lb. acting vertically downward.

9. Equilibrium.—If the resultant of forces acting on a body in the same straight line is zero, they are in *equilibrium.* The body on which these forces act will not move or if in motion will continue to move with the same speed. This condition of equilibrium might be stated in another way by saying that the sum of the forces in one direction must be equal to the sum of the forces in the opposite direction. Direction is not limited to either the horizontal or the vertical but may make any angle with the horizontal.

10. Action and Reaction.—To every *action,* there is an equal and opposite *reaction* (Newton's third law).

When one pushes against a wall, one exerts an *action* or *active force.* The wall presses against the hand with an equal and opposite force called a *reaction.*

When a weight is suspended from a hook, the weight exerts an action downward. The hook exerts an equal, opposite force called the reaction.

A load hanging from the center of a beam exerts a downward action that is opposed by an opposite reaction from the fibers of the beam.

The pressure of steam on the piston is an acting force or action. It is opposed by the reaction of the piston rod.

The pressure of the water against the dam is considered an action that is opposed by the reaction of the dam. The pressure of the wind against a signboard constitutes an action. The signboard exerts an equal, opposite force called a reaction.

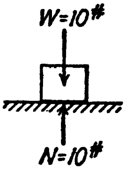


FIG. 10.

A body weighing 10 lb. rests on a table. The active force is the pull downward due to the earth's attraction. The table exerts an upward reaction N (Fig. 10).

Since the body does not move, the forces are in equilibrium. Then

$$R = N - 10 = 0$$

or

$$N = 10 \text{ lb.}$$

11. Tension and Compression Stress.—When two equal and opposite forces P act on a body such as a bar or rope,

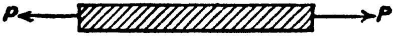


FIG. 11.

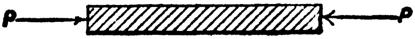


FIG. 11*a*.

as in Fig. 11, the body is in *tension*, because the tendency of the forces P is to stretch or pull the bar apart. A hoisting rope or cable is a good illustration of this action.

If the forces act in the opposite direction, as in Fig. 11*a*, the body is in *compression*, because the tendency of the forces P is to shorten, compress, or crush the body. This action is represented by a prop, brace, or column.

Two-force Piece.—When a bar is acted upon by two opposite forces that are in equilibrium, the bar is called a *two-force piece. It is always either in tension or compression, and the direction of the external forces is along the axis of the bar.*

The forces are termed *axial forces* and the corresponding stresses are called *axial stresses.*

This principle or definition should be thoroughly understood by the student, for it fixes the direction of the force acting on a two-force piece and, in the solution of problems in which such cases arise, the magnitude of the force is the only unknown quantity to be determined.

For a simple experiment in tension, a rope of any convenient length was chosen, and a force of 50 lb. applied at each end as in Fig. 12.

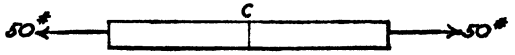


FIG. 12.

The rope was cut at any point *C*, and two spring balances fastened to the ends and hooked together (Fig. 13).

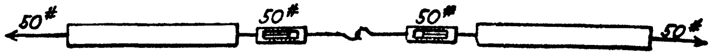


FIG. 13.

Each balance then read 50 lb., an indication that a force of 50 lb. was required to hold the parts of the rope together. This is just the force that was exerted by the fibers of the rope before it was cut.

12. Stress.—*The forces exerted by the fibers of a body to keep from being pulled apart when in tension or crushed when in compression are called "internal stresses" or simply "stresses."* The stress is considered as acting uniformly over the entire cross section of the body; it is therefore a uniformly distributed force.

In Fig. 14, the weight *B* of 100 lb. is suspended from the point *A* by the rope *AB*. *AB* is then in tension. Figure 14*a* shows block *A* resting on block *B*. The fibers of *B* are in compression owing to the weight *A*. In Fig. 2, the connecting rod *EC* and the piston rod *CD* are each in compression during the forward stroke as shown, and each will be in tension during the return stroke.

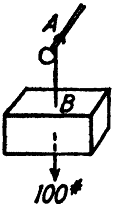


FIG. 14.

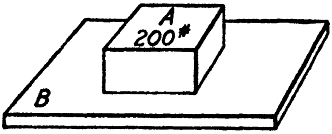


FIG. 14*a*.

If the total internal stress P exerted by the fibers at any cross section of a two-force piece is known and the area of the cross section is A, the stress per square unit of area is found by dividing the force by the area.

$$\textbf{Unit stress} = \frac{\textit{total force}}{\text{area resisting force}}$$

Let s be the stress per square inch. Then

$$s = \frac{P}{A} \tag{1}$$

where P is in pounds and A in square inches. The term *stress* will be used to indicate *unit stress* or pounds per square inch, *pounds per square foot, tons per square foot, grams per square centimeter, or other similar designations.* The most common designation is pounds per square inch.

The term *total stress* will be used to indicate the *total internal force.*

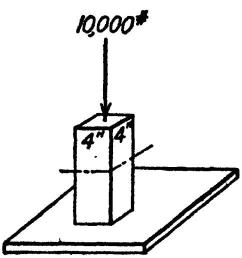


FIG. 15.

Example 1.—A 4- by 4-in. bar is under a compression force of 10,000 lb. Find the stress (Fig. 15).

Solution.—$P = 10{,}000$ $A = 4 \times 4$ in. $= 16$ sq. in.

$$s = \frac{10{,}000}{16} = 625 \text{ lb./sq. in.}$$

Example 2.—A steel rod 1 in. in diameter is subjected to a pull of 6,000 lb. What is the stress?

Solution.—$P = 6{,}000 \qquad A = \frac{\pi 1^2}{4} = 0.7854$

$$s = \frac{6{,}000}{0.7854} = 7{,}630 \text{ lb./sq. in. } \textit{Ans.}$$

13. Tensile and Root Area of Bolts.—Figure 16 shows a common type of bolt used to fasten the pieces of a structure. If it is in tension, the full cross section *AB* of the bolt does not have the greatest stress. The load on the nut is carried through the threads of the nut and bolt to the portion of the bolt on which the threads are cut. This area, which is smaller than the area of the bolt proper, is called the *root area.*

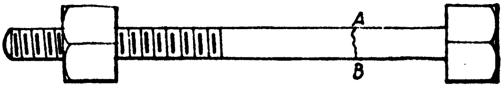


FIG. 16.

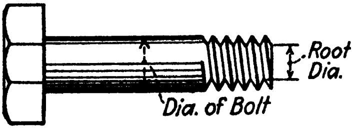


FIG. 17.

The stress across the root area is larger than at any other section and is the one to use in figuring the strength of the bolt (Fig. 17).

Handbooks of steel companies give details of the commercial-sized bolts, one such detail being the root area. For example, a 1-in. rod has a cross section of 0.7854 sq. in. but a root area of 0.551 sq. in.

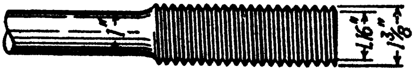


FIG. 18.

Material would be wasted by using long tie rods made like bolts. Rods with enlarged ends or upset ends are used, on which the threads are cut. As a result the root area is left the same as the main cross section (Fig. 18).

Example.—A 1-in. bolt carries a tensile load of 6,000 lb. Find the stress in the bolt.

Solution.—$P = 6{,}000$ lb. $A = 0.551$ (root area)

$$\therefore s = \frac{6{,}000}{0.551} = 10{,}900 \text{ lb./sq. in.}$$

14. Shear.—Figure 19 shows a piece of timber from which the portion to the right of the plane *BCF* and above the plane *BCH* has been removed. A distributed force, the resultant of which is represented by *P*, is applied to the face *BCF*, tending to move the upper portion *BE* to the left. An equal force *P* acting to the right on the lower part *AMH* resists the motion. The fibers of the beam on the face *ABCD* resist the motion of the upper and lower block.

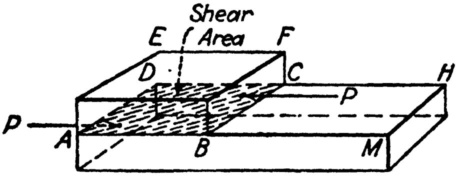


FIG. 19.

When two forces act on a body in opposite directions in parallel adjacent planes, they are called *shearing forces*, because the action of the forces is like the blades of a pair of shears. The fibers of the body in the plane along which the two portions tend to slide are in *shear*. The magnitude of this shearing force per square inch is called *shearing stress*.

It is to be understood that the planes in which the two forces *P* act are very close together or adjacent. The shear is then called *direct shear*.

If the forces *P* are some distance apart, a stress due to bending is developed. This would produce indirect shear, which will be discussed in a later chapter.

Example.—In Fig. (19), what will be the shearing stress on the area *ABCD*? $P = 2{,}000$ lb., $AB = 8$ in., and $BC = 4$ in.

Solution.—$P = 2{,}000$ lb. $A = 32$ sq. in.

$$s = \frac{2{,}000}{32} = 62.5 \text{ lb./sq. in.}$$

15. Shear in a Bolt.—Suppose two pieces of timber are united by a bolt of diameter *d* as shown in Fig. 20. Each of the pieces is in tension as shown. There is a tendency for the top timber to move over the bottom piece, and shearing stress is developed in the bolt across the section *CD*.

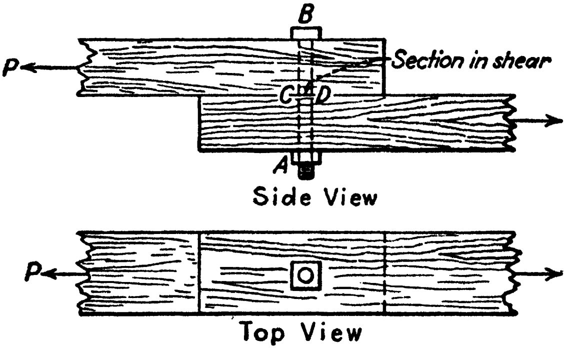


FIG. 20.

The bolt is in *single shear* because the shearing force acts on one section of the bolt.

The shearing stress is

$$s = \frac{P}{0.7854d^2} \tag{2}$$

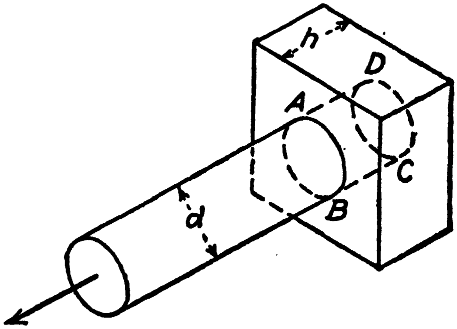


FIG. 21.

If a weight is suspended from a support by means of a bolt, the tensile load is carried to the head of the bolt, tending to shear it from the body of the bolt. The portion resisting the shear is the convex surface of a cylinder such as *ABCD* of Fig. 21. Now if *h* is the depth of the head and d the diameter of the body of the bolt, the shearing area

$$A = \pi dh$$

Then the shearing stress is found by Eq. (1)

$$s = \frac{P}{A} = \frac{P}{\pi dh} \qquad (3)$$

Example.—A 1-in. bolt has a head $\frac{5}{8}$ in. deep. It carries a load of 5,000 lb. Find the shearing stress on the head.

Solution.—$P = 5{,}000$, $d = 1$, $h = \frac{5}{8}$.

Then

$$s = \frac{5{,}000}{\pi \times 1 \times \frac{5}{8}} = \frac{8{,}000}{\pi} = 2{,}547 \text{ lb./sq. in.}$$

When three plates are connected, as in Fig. 22, it is assumed that one-half of the 2,000 lb. is resisted by the section AB of the rivet and the other half by CD. There are then two pairs of shearing forces acting on the rivet, which is said to be in *double shear*. The assumption is made that a rivet or pin in double shear is twice as strong as in single shear.

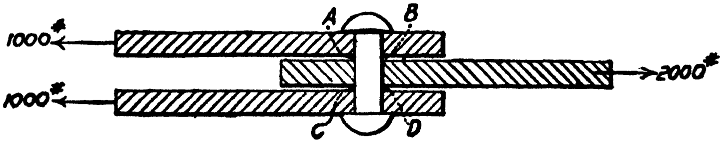


FIG. 22.

If a rivet or pin in double shear should fail by shear, it would appear as in Fig. 23.

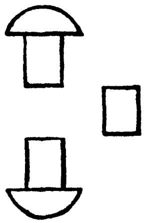


FIG. 23.

16. Safe Working Stress.—A *safe working stress* is a maximum load per square inch that is considered safe for pieces of iron, steel, wood, stone, concrete, etc., to carry. The values used in any case depend upon the uses to which the building materials or parts of a machine are to be put. These values are decided upon by builders and engineers, through years of experience and experiment.

The following table contains safe values of some of the most used materials. These numbers are only average

TABLE I

Material	Tension	Compression		Shear
		Perpendicular to grain	Parallel to grain	
Steel structural	16,000–18,000	16,000		10,000
rivet	16,000	18,000 (bearing)		8,800
Wrought iron	12,000	12,000		9,000
Cast iron	3,000	12,000		2,500
Pine, southern[1]				
Fir, Douglas[1]				
Brickwork (in lime)	......	110		
Brickwork (Portland cement)	......	250		
Concrete (Portland cement)	......	350		

NOTE.—For more specific values, see steel handbooks or building specifications.

[1] See Appendix, Table XVII, for the safe working stresses for all kinds and grades of wood, under different working conditions.

values for the solution of problems. Just what value is used by the designer depends upon the material and the manner of loading and often upon local conditions.

Example 1.—Find the safe load that a ¾-in. bolt will carry in single shear; in double shear. If the bolt has a ⅝-in. head, what safe load would the bolt carry in tension?

Solution.—From the table, a safe shearing stress on a bolt in structural work is 10,000 lb. per square inch, and 16,000 lb. in tension. Now the area of a ¾-in. bolt is

$$A = \frac{\pi}{4}\left(\frac{3}{4}\right)^2 = 0.44 \text{ sq. in.}$$

For single shear,

$$P = sA = 0.44 \times 10{,}000 = 4{,}400 \text{ lb.}$$

For double shear

$$P = 2 \times sA = 8{,}800 \text{ lb.}$$

From formula (3), the shearing area on the head is

$$A = \pi \times \tfrac{3}{4} \times h = \pi \times \tfrac{3}{4} \times \tfrac{5}{8} = 1.47 \text{ sq. in.}$$

Then

$$P = sA = 10{,}000 \times 1.47 = 14{,}700 \text{ lb.}$$

The root area of a ¾-in. bolt is 0.302.
Then

$$P = sA = 16{,}000 \times 0.302 = 4{,}830 \text{ lb.}$$

From these results, it is evident that the depth of the head is greater than necessary for the tensile load that the bolt will carry, since the head will carry 14,700 lb. whereas the bolt will only carry 4,830 lb.

Example 2.—Answer the questions of the preceding example for ½-, ⅝-, ⅞-, and 1-in. bolts, using a handbook for root area and depth of head.

Example 3.—A generator weighing 2,100 lb. is suspended by a wrought-iron bolt. Find the size of bolt necessary to carry the load.

Solution.—From Table I, $s = 12{,}000$. Also, $P = 2{,}100$ lb.

$$\therefore A = \frac{P}{s} = \frac{2{,}100}{12{,}000} = 0.175 \text{ sq. in.}$$

But this is root area. From the handbook the nearest commercial root area is 0.202, which corresponds to a ⅝-in. bolt.

Example 4.—If the cylinder of Fig. 2 is 12 in. in diameter and the steam pressure is 100 lb. per square inch, find the necessary diameter of the piston rod, if 7,000 lb. is a safe working stress.

Solution.—The area of the cylinder is $(\pi/4) \times (12)^2 = 113.1$ sq. in. Then the steam pressure is $100 \times 113.1 = 11{,}310$ lb.

Let d = the diameter of the piston rod. Then its area is $(\pi/4)d^2$. Since 7,000 lb. per sq. in. is the safe stress,

$$\frac{\pi}{4}d^2\, 7{,}000$$

is the load on the piston rod.

$$\frac{\pi}{4}d^2\, 7{,}000 = 11{,}310 \text{ lb.}$$

$$d^2 = \frac{11{,}310 \times 4}{7{,}000\pi} = 2.05$$

$$\therefore d = 1.43 \text{ in.}$$

Table I is based on the assumption that the structure carries a dead load, such as the weights of the parts, load, furniture, or machines fixed in place. But in the case of a bridge with moving loads that produce varying stresses or in machine parts such as the connecting rod of an engine where the stress changes from tension to compression each revolution, the safe working stress is much lower. What constitutes a safe stress is a matter largely for the judgment of the designer, who must or should know the kinds of loading and whether or not the materials are to be thoroughly tested and inspected.

17. Ultimate Strength of Materials.—The load in pounds per square inch that will rupture a specimen is called its *ultimate strength.* The three stresses are referred to as *ultimate tensile strength, ultimate compression strength,* and *ultimate shearing strength.*

TABLE II

Material	Tension	Compression	Shear
Steel, hard	100,000	120,000	50,000
Steel, structural	60,000	60,000	50,000
Nickel steel	90,000	60,000	
Chrome steel	135,000	115,000	
Monel metal	75,000		
Duralumin	50,000		
Wrought iron	50,000	50,000	40,000
Cast iron	20,000	90,000	20,000
Timber (along grain)[1]			
Stone	....	6,000	1,500
Brick	....	3,000	1,000
Concrete	250	2,500	

[1] See Appendix, Table XVII.

Table II gives values at which specimens of materials have been found to rupture, under a dead or steady load.

If the ultimate strength of a material is divided by the safe working stress, the quotient is called the *factor of safety.*

If Tables I and II are compared, it is seen that the factor of safety for steel and iron is 4, for timber about 10, and for stone from 15 to 20.

Table III contains factors of safety for the three types of loading. These are only averages and seem near those in common use. They will be used to guide the student in solving problems in this text.

18. Selection of Safe Working Stress.—The general tendency in good practice is to select the safe working stresses with reference to the elastic limit, it being made about one-half of the value of the stress at the elastic limit. This procedure seems to be a logical one, for it is desirable never to overstress to the extent of causing a permanent deformation. Since the working stress is taken as about one-half the elastic limit, it is evident that the member may be overstressed almost 100 per cent without causing a permanent deformation of the member.

TABLE III

Material	Steady load stress	Varying load stress	Repeated or reversed load stress
Steel, hard	5	8	15
Steel, structural	4	6	10
Iron, wrought	4	6	10
Iron, cast	7	10	20
Timber[1]			
Brick and stone	15 to 20	25	30

[1] See Appendix, Table XVII.

Example 1.—A block of wood 8 × 10 in. standing on end supports a load of 45,000 lb. Find the factor of safety in compression.

Solution.—$P = 45{,}000$ lb. $A = 8 \times 10 = 80$ sq. in.

$$s = \frac{45{,}000}{80} = 562.5 \text{ lb./sq. in.}$$

Assume the ultimate compressive strength of timber is 8,000 lb./sq. in.

$$\therefore \frac{8{,}000}{562.5} = 14.2$$

or, say 15 for the factor of safety.

Example 2.—Figure 24 shows a steel rod under a tensile load of 30,000 lb. If the diameter of the hole in the washer is 2 in., find the diameter of the washer.

Solution.—The 30,000 lb. is carried through the head of the bolt to the washer, and then to the wood. Now, if a safe compressive stress is 1,000 lb., the necessary bearing area on the wood is

$$\frac{30{,}000}{1{,}000} = 30 \text{ sq. in.}$$

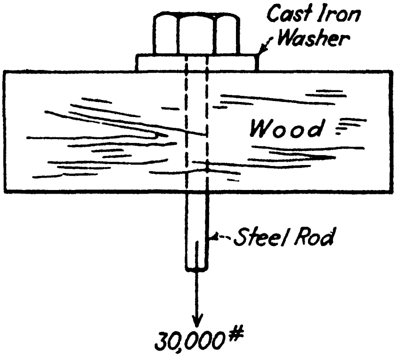


FIG. 24.

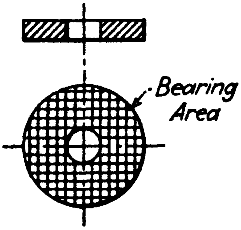


FIG. 24*a*.

But this is the difference in area between a large circle of diameter *d* and one of diameter 2 in. Then

$$\frac{\pi}{4}d^2 - \frac{\pi}{4}(2)^2 = 30$$

$$d^2 = \frac{4}{\pi} \times 30 + 4 = 42.2$$

$$\therefore d = 6.5 \text{ in. (see Fig. 24}a\text{)}$$

Example 3.—A steel rod 1 in. in diameter and 20 ft. long is suspended vertically. Find (*a*) the reaction at the point of suspension, (*b*) the stress in the rod at distances of 5 ft. and 10 ft. from the top, respectively. (Steel weighs 490 lb. per cu. ft.)

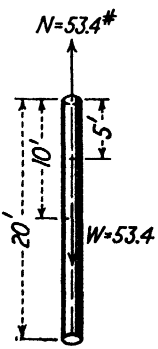


FIG. 25.

Solution.

a. The volume of the rod (Fig. 25) is

$$V = \frac{(\pi/4) \times 1^2 \times 20}{144} = \frac{5\pi}{144} \text{ cu. ft.}$$

$$\text{Weight} = \frac{5\pi}{144} \times 490 = 53.4 \text{ lb.}$$

Since the reaction N at the top is in equilibrium with the weight, they are equal and opposite forces.

$$\therefore N = 53.4 \text{ lb.}$$

b. The fibers of the rod across a section C (Fig. 26) 5 ft. from the top must support 15 ft. of rod below the section. Since 20 ft., of rod weighs 53.4 lb., 15 ft. will weigh $\frac{15}{20} \times 53.4 = 40$ lb. If P = total fiber stress, then, from the conditions of equilibrium, $P = 40$.

But the area $A = (\pi/4)\,(1)^2 = 0.7854$ sq. in.

$$\therefore s = \frac{40}{0.7854} = 51 \text{ lb./sq. in.}$$

c. The fibers at the middle section B support 10 ft. of rod that weighs $\frac{1}{2} \times 53.4 = 26.7$ lb.

$$\therefore s = \frac{26.7}{0.7854} = 34 \text{ lb./sq. in.}$$

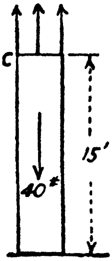


Fig. 26.

19. Stresses in Thin Pipes or Cylinders.—When a thin cylindrical shell, such as a boiler, is subjected to steam pressure or water pipes are subjected to the pressure of the water, the forces tending to rupture the cylinder act normally to the surface of the pipe. Internal stresses are developed in the metal of the cylinder which must not exceed the safe working stress of the metal if the pipe is to be safe. These internal pressures tend to rupture the cylinder in two ways: first, along seams parallel to the elements of the cylinder and, second, along a seam corresponding to a circumference of the cylinder. Figure 27 shows a longitudinal section $ABCD$ with the normal pressure p of the steam or fluid on the curved surface of the cylinder. Suppose that a flat cover $ABCD$ is added to Fig. 27 so that the pipe or boiler will be a half cylinder as shown in the top view of Fig. 27a. Since the fluid pressure is exerted in a direction normal to the surface, it may be said in accordance with the principle that action is equal and opposite to reaction, that the total force tending to tear the curved surface from the flat surface is equal and opposite to the

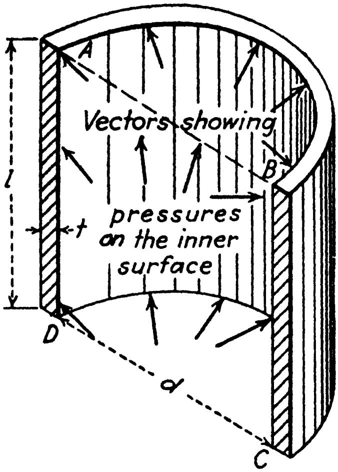


Fig. 27.

force which would tend to tear the flat surface from the curved surface. But the area of the flat surface is equal to the projection of the curved surface on the plane of the flat surface. A rule may be stated as follows, thus:

The resultant pressure is the pressure per square inch of the fluid times the projection of the curved surface on the plane determined by a diameter and the longitudinal axis of the cylinder. But this projection is equal to *ld*, where *l* is the length of the pipe in inches and *d* the diameter of the pipe in inches.

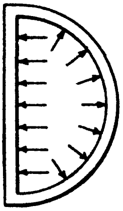


FIG. 27*a*.

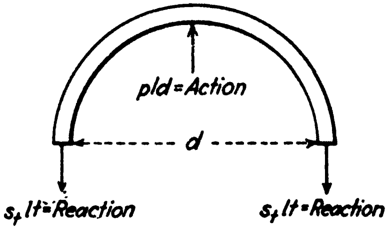


FIG. 27*b*.

Then

$$R = pld$$

But R is resisted by the tensile stress in the two sections of metal of length l and thickness t. Let s be the stress per square inch.

$$2slt = pld$$

or

$$s = \frac{pd}{2t} = \frac{p2r}{2t} = \frac{pr}{t} \tag{4}$$

Figure 27*b* shows the two sets of forces that must be in equilibrium.

Example 1.—Find the stress per square inch in a 24-in. water main, the plate being ⅜ in. thick and the water pressure 90 lb. per sq. in.

Solution.—$p = 90$, $r = 12$, and $t = \frac{3}{8}$. Substituting in Eq. (4), we have

$$s = \frac{90 \times 12}{\frac{3}{8}} = 2{,}880 \text{ lb./sq. in.}$$

Example 2.—What water pressure will probably burst a 16-in. water pipe $\frac{3}{8}$ in. thick?

Solution.—Use s = 64,000 lb. Substituting in Eq. (4) produces

$$64{,}000 = \frac{p \times 8}{\frac{3}{8}}$$
$$p = 3{,}000 \text{ lb./sq. in.}$$

20. Longitudinal Tension.—The end pressure on the *pipe* is $p \times \pi r^2$. This is resisted by the fiber stresses on the cross section of the pipe. The area of the section is approximately $2\pi rt$, from Fig. 28. If s is the resisting tensile stress, then

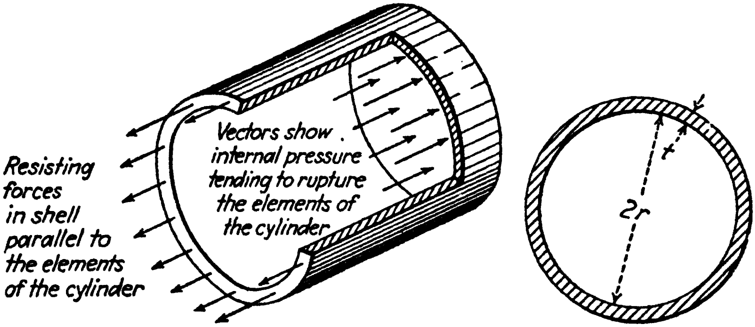

Resisting forces in shell parallel to the elements of the cylinder
Vectors show internal pressure tending to rupture the elements of the cylinder
t
2r

FIG. 28.

$$s2\pi rt = p\pi r^2$$
$$s = \frac{pr}{2t} = \frac{pd}{4t} \tag{5}$$

This equation is also true for thin spheres, since $\pi r^2 p$ is the resultant pressure on the projected surface which is the great circle of the sphere, as further explained below.

21. Stresses in Hollow Spheres.—The force tending to tear a hollow sphere along a great circle may be computed as the longitudinal tension of the cylindrical tank or pipe is computed. The force is equal to the product of the internal pressure multiplied by the area of a great circle. This area is equal to the projection of the surface of a hemisphere on the plane of a diameter. Thus, in Fig. 29 the pressure is applied normal to the interior surface. If the flat cover AB is placed on the hemisphere, then the total force normal to this surface will tend to pull the cover away from the hemisphere. Since action is equal to reaction, the force tending to pull the hemisphere away from the flat surface will be equal and opposite to the force just indicated. The amount of this force is equal to $\pi r^2 p$. A rule for obtaining this force may be stated as follows:

The resultant pressure or total force is equal to the pressure per square inch in the fluid or gas multiplied by the projection of the surface of the hemisphere on the plane of the diameter. But this projection is equal to the area of a circle or πr^2.

Then

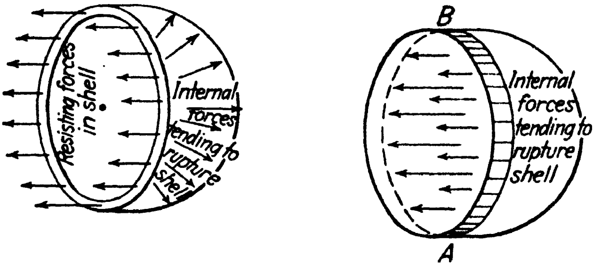

Resisting forces in shell
Internal forces tending to rupture shell
B
Internal forces tending to rupture shell
A

FIG. 29.

$$R = \pi r^2 p$$

But R is resisted by the tensile stress in the ring whose area is taken as $2\pi rt$. If s = stress per square inch, then

$$s2\pi rt = \pi r^2 p$$

Canceling, we have

$$s = \frac{pr}{2t} \tag{5}$$

Similarly, for the sphere, the resultant force acting on either half tends to pull it away from the other half. The amount of the force will be the same as for the hemisphere, and the stress produced along the great circle of the sphere will be

$$s = \frac{pr}{2t} \tag{5}$$

Example 1.—A closed hemispherical tank is 72 in. in diameter. It contains air under a pressure of 20 lb. per sq. in. What will be the stress in the shell just under the lid if the metal is $\frac{3}{8}$ in. thick?

Solution.—The area of the metal resisting the force is

$$2\pi rt = 2\pi 36 \times \tfrac{3}{8} = \pi 27 \text{ sq. in.}$$

The force $R = \pi 36^2 \times 20$. Then

$$s\pi 27 = \pi 36^2 \times 20$$

$$s = \frac{36^2 \times 20}{27} = 960 \text{ lb./sq. in.}$$

This result can also be obtained by substituting directly in the formula, or

$$s = \frac{pr}{2t} = \frac{20 \times 36}{2 \times \frac{3}{8}} = 960 \text{ lb./sq. in.}$$

Example 2.—The pressure on the end of a 20-ft. cylindrical steel tank is 70 lb. per sq. in. If a safe stress is 16,000 lb. per sq. in., find the necessary thickness of plate.

Solution.—$s = 16{,}000$, $p = 70$, and $r = 120$ in. Substituting in Eq. (5) for the end pressure, we have

$$t = \frac{70 \times 120}{2 \times 16{,}000} = 0.26 \text{ in.}$$

For side pressure, substitute in Eq. (4).

$$t = \frac{70 \times 120}{16{,}000} = 0.52 \text{ in.}$$

i.e., a thickness twice as great is needed for side pressure as for the end pressure. Comparing Eqs. (4) and (5) verifies the result in any case.

22. Pressure of Water in a Tank.—A fundamental theorem in hydrostatics states that at any point in a body of liquid the intensity of the pressure due to the weight of the liquid is the same in all directions. This pressure is equal in pounds per square inch to the weight of a column of water 1 in. square and of height equal to the depth of the point below the surface of the water.

A cubic foot of water weighs 62.5 lb. Then, if the point is h ft. below the surface, the pressure p at that point is

$$p = \frac{h}{144} \times 62.5 = 0.434h$$

where h is in feet and p is in pounds per square inch.

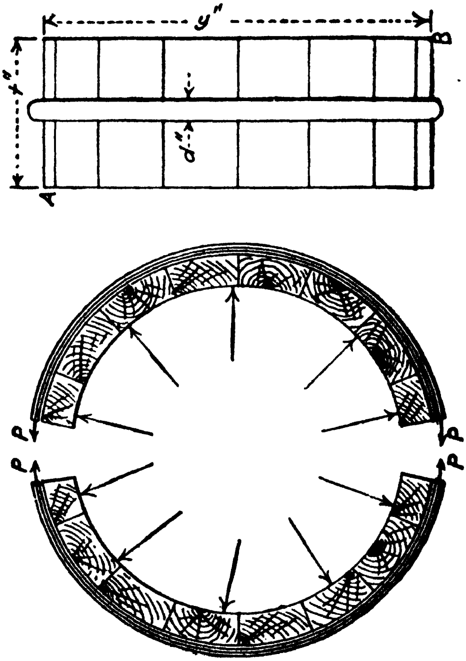


FIG. 30.

When $h = 10$ ft., $p = 4.34$ lb.; $h = 20$ ft., $p = 8.68$ lb., etc.

If a wood-stave tank is supported by round iron hoops, d in. in diameter, the height of tank that each hoop will support is found as follows:

Suppose the tank is y in. in diameter. Let t be the height of tank to be supported by the hoop as shown in Fig. 30.

Then, if s is a safe working tensile stress for the hoop,

$$2P = 2\frac{\pi d^2}{4}s$$

The pressure of the water against the rectangle AB is

$$pyt = 0.434hyt$$

$$\therefore \frac{2\pi d^2}{4}s = 0.434hyt$$

or

$$t = \frac{\pi d^2 s}{0.868hy}$$

where all dimensions are in inches, except h, which is in feet.

Example.—A tank 13 ft. in diameter and 14 ft. deep is hooped with ⅝-in. round wrought-iron rods. Find the necessary spacing.

Solution.—$d = \frac{5}{8}$, $s = 10{,}000$, $y = 156$ in. At the bottom of the tank, $h = 14$ ft. Then

$$t = \frac{\pi(\frac{5}{8})^2 \times 10{,}000}{0.868 \times 14 \times 156} = 6.47 \text{ in.}$$

Then, at the bottom of the tank, the hoops should not be farther apart than 6.47 in.

When

$h = 13$ ft., $t = 6.97$ in.; $h = 12$ ft., $t = 7.57$ in.
$h = 11$ ft., $t = 8.24$ in.; $h = 10$ ft., $t = 9.06$ in.
$h = 8$ ft., $t = 11.33$ in.; $h = 6$ ft., $t = 15.1$ in.
$h = 5$ ft., $t = 18.12$ in.; $h = 4$ ft., $t = 22.7$ in., etc.

To be entirely correct the depth h should be taken to the middle of the strip of width t. However, by taking h to the bottom of the strip, the force involved in the computations is slightly larger than the actual force and the design is always on the side of safety.

In practice, it would not be necessary to calculate the values of t for every foot. Solve for t, say at 14 ft., at 10 ft., and at 5 ft. The hoops may be spaced from these results, care being taken that each space is not greater than the calculated value of *t*. Figures 31 and 32 show the spacings and tank with the hoops in place.

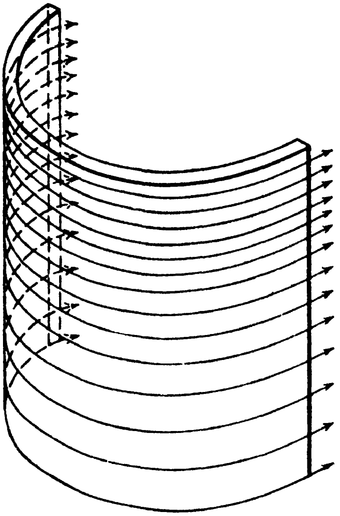


FIG. 31.

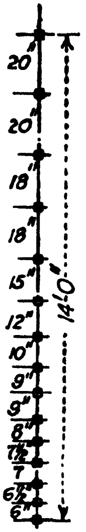


FIG. 32.

Problems

1. A bar of steel 1 by 1½ in. in cross section is subjected to a pull of 6,000 lb. What is the unit tensile stress in the bar?

2. A short prism of steel 2 in. in diameter supports a load of 2,000 lb. What is the unit compressive stress?

3. A steel rod 1¼ in. in diameter and 60 ft. long is suspended vertically. What is the stress in the rod due to its own weight? (Steel weighs 490 lb. per cu. ft.)

4. A structure weighs 200 tons. Its weight is equally supported by 24 steel posts, each having a cross section of 20 sq. in. What is the unit compressive stress in each post?

5. In Prob. 4, what should be the cross-sectional area of each post, if the maximum unit stress is not to exceed 750 lb. per sq. in.?

6. A horizontal beam weighing 300 lb. and carrying a distributed load of 3,000 lb. is supported by two ¾-in. bolts at each end of the beam. If the load on the beam subjects the bolts to a direct tension, what is the unit stress in the root area?

7. A 1-in. steel bolt with a head ⅞ in. deep is under a tension of 10,000 lb. What is the shearing stress tending to shear off the head (see Fig. 33)?

8. A platform 16 by 24 ft. is to support a uniform load of 200 lb. per sq. ft. of surface. It rests on twelve 8- by 8-in. posts so spaced that each post carries an equal amount of load. What is the unit stress in each post? Show how you would arrange the posts.

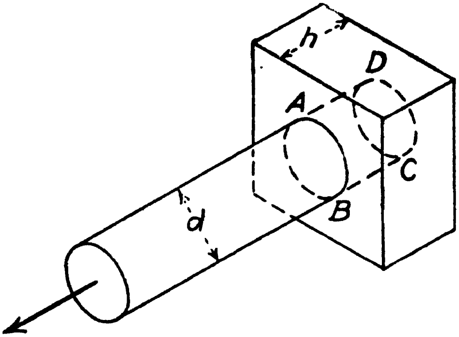


FIG. 33.

9. What will be the unit tensile and shearing stresses in a 1-in. steel bolt under a tensile load of 10,000 lb.? The head and the nut are both 1 in. in height. The root area of the bolt is 0.55 sq. in.

10. In Prob. 7, what should be the depth of the head in order to have the bolt equally strong in tension and in shear?

11. A wrought-iron bar ½ in. square fails under a pull of 12,000 lb. What is the ultimate strength of the iron?

12. A short post of Douglas fir is 6 in. in diameter. What load will it support if a factor of safety of 8 is to be used? What will be the stress if the post supports a load of 30,000 lb.?

13. What will be the unit tensile and shearing stresses in a 1-in. steel bolt with a head ¾ in. deep, if the bolt is subjected to a pull of 12,000 lb.?

14. A pull of 25,000 lb. is applied to a steel bolt 1¼ in. in diameter, and with a head 1 in. deep. What are the unit tensile and shearing stresses? The root area = 0.89 sq. in.

15. Compute the length of a structural steel rod suspended at the upper end so that the stress at a point 100 ft. from the upper end will be equal to one-half that at the elastic limit.

16. In Prob. 15, what should be the length if the stress at the upper end is to equal the elastic limit?

17. What height of concrete wall would cause a stress of 100 lb. per sq. in. at the base?

18. What height of brick wall would cause a stress of 100 lb. per sq. in. at the base?

19. The lower one-third of a wall consists of concrete, and the upper two-thirds consists of brick. What total height of wall would cause a stress of 100 lb. per sq. in. at the base?

20. In Fig. 34, a load of 36,000 lb. is supported by an 8- by 8-in. wooden post. The allowable pressure between the plate and the footing is 300 lb. per sq. in. The allowable pressure on the soil shall not exceed 5,000 lb. per sq. ft. What should be the area of the plate and of the footing? Assume that the footing contains 8 cu. yd. of concrete.

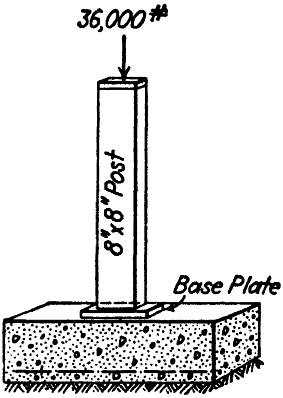


FIG. 34.

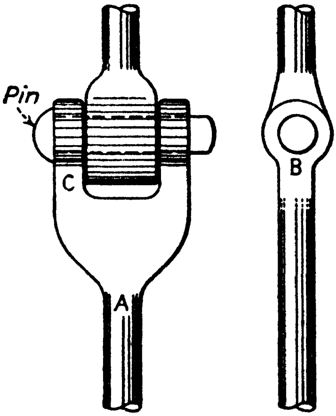


FIG. 35.

21. Find the dimensions *A*, *B*, *C* for the cast-iron yoke of Fig. 35 so that it will be strong enough in tension to shear a 1-in. hard-steel pin.

22. The steam pressure in a cylinder of a pumping engine is 100 lb. per sq. in. The cylinder is 14 in. in diameter. The cylinder head is held by 16 steel bolts. If the initial pull on each bolt is 2,000 lb., what will be the fiber stress in the bolts? The root area of the bolts = 0.55 sq. in.

23. Find the dimensions of the various parts in Fig. 34 if the load is 42,000 lb.

24. In Fig. 36, what are the areas needed at all the bearing surfaces between the various parts of the footing? The pedestal weighs 2,000 lb. The concrete footing contains 5 cu. yd. The allowable soil pressure is 8,000 lb. per sq. ft. The bearing on the concrete shall not exceed 500 lb. per sq. in., and the allowable bearing of the column on the cast-iron pedestal is 10,000 lb. per sq. in.

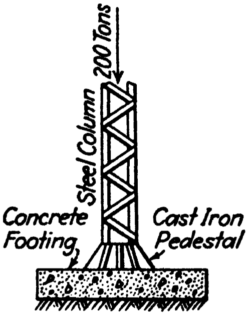


FIG. 36.

25. In Fig. 37, what must be the diameter of the hard-steel pin so that with a pull applied to the vertical shaft the fastening is equally strong in tension and in shear? All parts are made of hard steel.

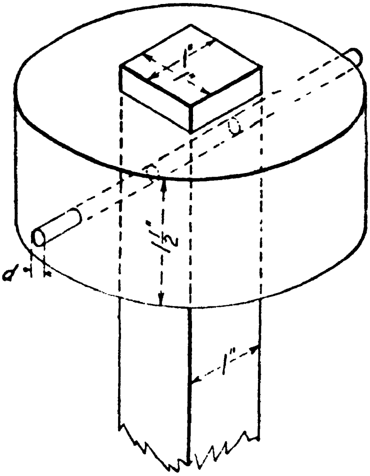


FIG. 37.

26. What force is required to punch a hole 1 in. in diameter through a wrought-iron plate ½ in. thick?

27. A tank 8 ft. in diameter and 6 ft. deep is hooped with ½-in. round wrought-iron rods. What must be the spacing of the rods?

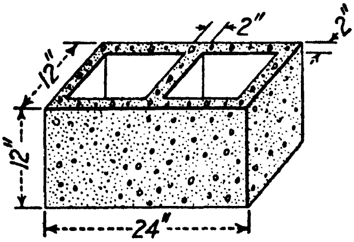


FIG. 38.

28. In Fig. 38, what safe load would the building block support?

29. What safe internal pressure should be used for a steel water pipe 6 in. in diameter and with a ½-in. shell?

30. What should be the thickness of a plate for a 10-in. steam pipe to resist a pressure of 80 lb. per sq. in.? The fiber stress in the pipe shall not exceed 9,000 lb. per sq. in.

31. What pressure may be used in a 36-in. cast-iron water pipe with 2-in. walls, a factor of safety of 10 being allowed?

32. In Prob. 22, what should be the diameter of a steel piston rod if a factor of safety of 6 is used?

33. The steam pressure in a cylinder with 12 in. inside diameter is 150 lb. per sq. in. The cylinder head is held on by 12 steel bolts equally spaced around the circumference. What should be the dimensions of the bolts and heads with a factor of safety of 8.

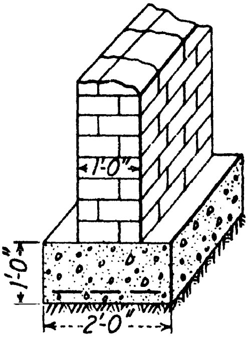


FIG. 39.

34. What height of brick-masonry wall may be built on the concrete footing shown in Fig. 39 so that the soil pressure may not be in excess of 1 ton per sq. ft.?

35. What will be the tensile unit stress in the shell of a spherical tank 20 ft. in diameter and containing gas under a pressure of 100 lb. per sq. in.? The shell is made of ½-in. steel plate.

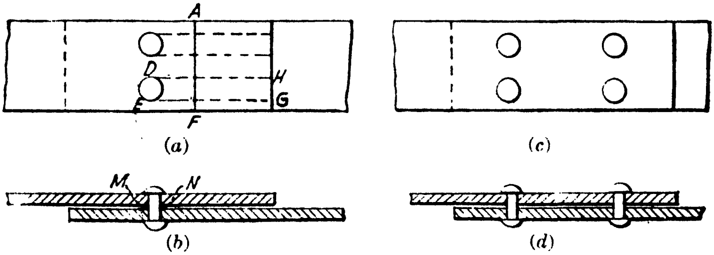
36. Find the necessary spacing of wrought-iron hoops, ¾ in. in diameter, for a wood-stave tank 20 ft. in diameter and 16 ft. deep. Assume s_t = 10,000 lb. per sq. in.

CHAPTER II

WELDED AND RIVETED JOINTS

23. Structural Connections.—In the construction of a boiler or in the erection of a building or bridge the several individual members or pieces must be securely connected. Such connections are obtained by means of welds or bolts and rivets and are designated respectively as welded joints and riveted joints.

Increasing skill in welding, both inside and outside the shop, has brought about a much enlarged use of the method in all types of tank, boiler, and structural work. The theory of welding is very simple and will be given in Art. 40.



(a) (c)

(b) (d)

FIG. 40.

24. Riveted Joints.—Riveted joints are designated either *lap* or *butt* joints. When two plates are placed as in Fig. 40 with the ends overlapping, they form a lap joint. Figures 40*a* and *b* represent the top and side view of a lap joint with a single row of rivets, and Figs. 40*c* and *d* represent one with a double row of rivets.

When two plates are placed end to end with cover plates above and below and are connected by two or more rows of rivets as in Fig. 41, they form a butt joint. Figures

41*a* and *b* represent a single-riveted butt joint, and Figs. 41*c* and *d* represent a double-riveted butt joint.

The actual stresses developed in riveted joints are quite complex, but in practice they are treated as simple tension, compression, and shearing stresses.

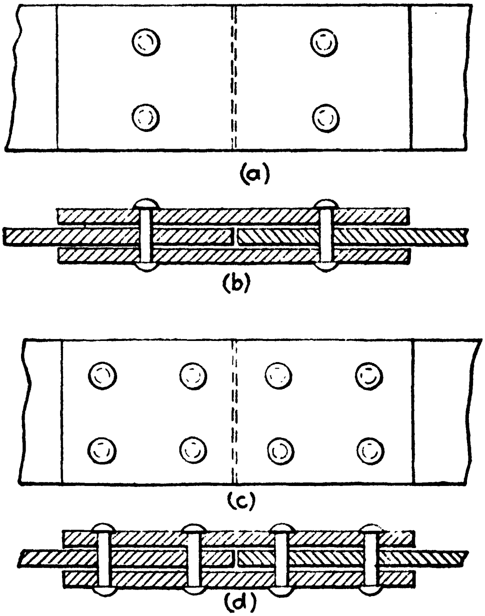


Fig. 41.

The analysis of riveted joints involves the continued use of Eq. 1 (Chap. I):

$$s = \frac{P}{A}$$

where s represents the particular stress to be determined. It also involves the basic general relation that action is equal to reaction.

25. Stresses in a Riveted Joint.—Two steel plates A and B are connected by two pins or rivets C and D (Fig. 42) to form a joint. Two equal and opposite forces P are applied to A and B, tending to slide the plate A to the left and B to the right. This motion is prevented by the shearing resistances of the cross sections of the rivets C and D in the planes of contact of the plates. Since there are two rivets, the pull on each is $P/2$ and not P as it may at first seem to be. A little study of the side view will show the true relation.

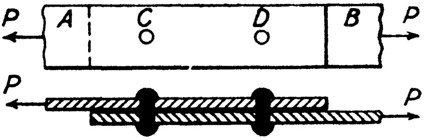


FIG. 42.

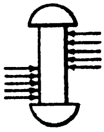


FIG. 42a.

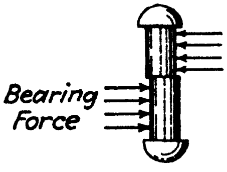


FIG. 42b.

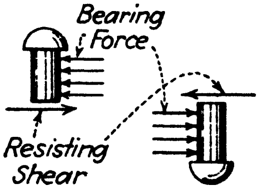


FIG. 42c.

Figure 42a shows the shearing forces acting on a rivet. Figure 42b shows the beginning of shear failure due to the action of the forces. Figure 42c shows the action of bearing and shearing forces.

If A is the area of the cross section of the rivet in square inches and n the number of rivets, the unit shearing stress s_s is obtained from the relation

$$s_s = \frac{P}{n} \div A = \frac{P}{nA} \qquad (6)$$

Example 1.—Let P = 5,000 lb. and the diameter of the rivets = 1 in.
Then

$$s_s = \frac{5{,}000}{2 \times (\pi/4)} = \frac{5{,}000}{1.5708} = 3{,}183 \text{ lb./sq. in.}$$

Example 2.—Two plates A and B each 12 in. wide and 1 in. thick are connected by four ⅞-in. rivets (Fig. 43). Two forces of 20,000 lb. are applied to A and B.

a. Find the shearing stress s_s on the rivets.

b. Find the tensile stress s_t in the section of the plate *HK*.

c. Find the tensile stress s_t in the section of the plate *CD*, through the center of the rivet holes.

d. Find s_t for the section *EF* between the two rows of rivets.

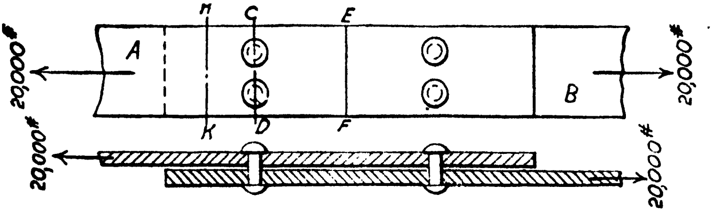


FIG. 43.

Solution.

a. Since the load $P = 20,000$ is carried by four rivets, each must hold one-fourth, or 5,000 lb.

$$\therefore s_s = \frac{5,000}{(\pi/4)(\frac{7}{8})^2} = \frac{5,000}{0.602} = 8,305 \text{ lb./sq. in.} \qquad \text{[Eq. (2)]}$$

b. The cross section of the plate is a rectangle 12 by 1 in. (Fig. 44). The entire load of 20,000 lb. is carried by the rectangle with area 12 sq. in.

$$\therefore s_t = \frac{20,000}{12} = 1,667 \text{ lb./sq. in.} \qquad \text{[Eq. (1)]}$$

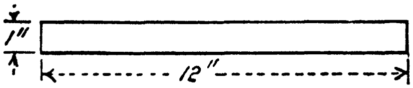


FIG. 44.

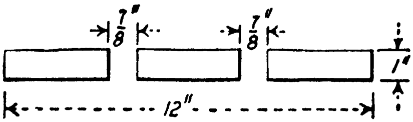


FIG. 45.

c. The section of the plate through *C* and *D* will cut rectangles from the rivets that are 1 in. deep and ⅞ in. wide (Fig. 45).

The area to carry the load is then

$$A = (12 - 2 \times \tfrac{7}{8}) \times 1 = 10.25 \text{ sq. in.}$$

$$\therefore s_t = \frac{20,000}{10.25} = 1,950 \text{ lb./sq. in.}$$

Figure 45*a* shows a top view of the tension forces acting through a section of the plate along a row of rivets.

d. Since the assumption is made that each rivet carries an equal part of the load, the two rivets through section *CD* will transmit one-half of the total load to the lower plate. There is then left but 10,000 lb. for the section *EF*, which has an area of 12 sq. in. Then

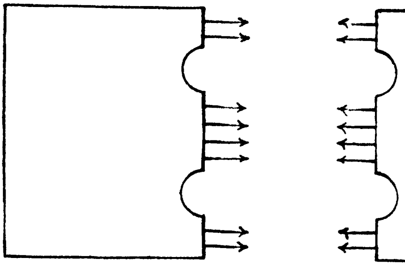


FIG. 45*a*.

$$s_t = \frac{10{,}000}{12} = 834 \text{ lb./sq. in.}$$

Since the section through the rivets has a higher unit stress than any other part of the plate, it is therefore the weakest section.

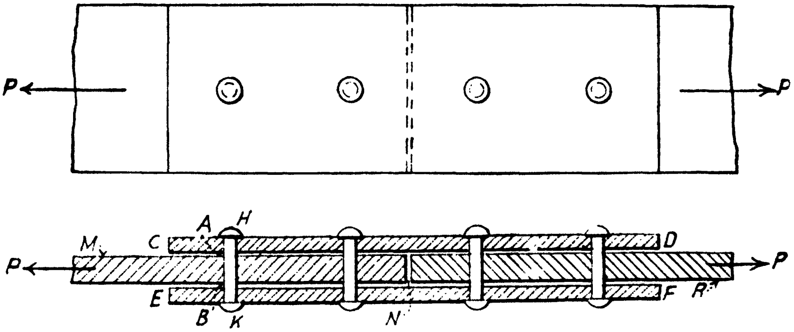


FIG. 46.

26. Single Shear.—When rivets connecting two plates, as in Fig. 43, resist shearing across one section, they are in single shear.

When there is a tendency to shear a rivet or bolt across two sections, it is in double shear. A rivet in double shear is twice as strong as one in single shear, because the area resisting the shear is twice as great.

27. Double Shear.—Another application of the principle of double shear occurs when two plates are placed end to end, with a cover plate on top and one on the bottom connected with the two middle, or main, plates by rivets (Fig. 46).

In this case the tendency of the force *P* is to pull the plates *MN* and *NR* from between the cover plates, or straps, *CD* and *EF*. This tendency to move is resisted by the rivets and cannot be fulfilled until all the rivets shear across two sections, such as *A* and *B* of rivet *HK*. If these rivets fail by shear, they will appear as in Fig. 47.

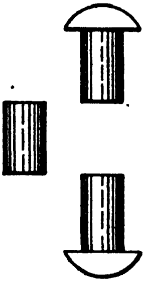


Fig. 47.

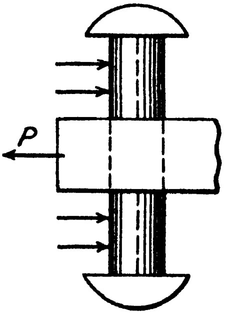


Fig. 47*a*.

Figure 47*a* shows the shearing forces on a rivet in double shear.

Example.—In Fig. 46, let $P = 20{,}000$ lb. and each of the rivets be ⅞ in. in diameter. Since there are two rivets in each plate to transmit the 20,000 lb. to the straps, each must carry one-half the load, or 10,000 lb.

The area of a section of the rivet is $\pi/4\ (\frac{7}{8})^2 = 0.602$ sq. in.

Being in double shear the area resisting shear is $2 \times 0.602 = 1.204$ sq. in.

$$\therefore\ s_s = \frac{10{,}000}{1.204} = 8{,}333 \text{ lb./sq. in., approximately.}$$

28. Bearing Stress.—With reference again to Fig. 46, it is evident that, when the force *P* is applied to the plates, each plate is forced against the curved surface of the rivet and tends to crush it and the plate. Thus a compressive force is produced which varies in intensity per square inch for various points on the curved surface. However, it is customary in figuring this bearing stress to take a section through the axis of the rivet, perpendicular to the direction of the applied forces. *This section is a rectangle of height t, the thickness of the plate, and width d, the diameter of the rivet.* Then the bearing area is

$$A = td$$

This bearing area is arrived at by assuming a similarity between the action of the forces applied to the curved surface of the rivet and the fluid pressure on the curved surface of a pipe or tank (see Art. 19).

If s_c is the compressive or bearing stress,

$$s_c = \frac{P}{td} \qquad (7)$$

Example.—Given $P = 20{,}000$, $t = 1$ in., $d = \frac{7}{8}$ in., to find the bearing stress.

Solution.—From Eq. (7), $s_c = \frac{P}{td} = \frac{20{,}000}{1 \times \frac{7}{8}} = 22{,}857$ lb./sq. in.

29. Failure of Riveted Joints.—A riveted joint may fail owing to one of the plates pulling apart along any cross section. This action is called "failure in tension" or "tearing of plates." Or it may fail because of crushing of the rivets or of the metal in front of the rivets. This type is called "failure in bearing" or "failure in compression." Another cause for failure in a riveted joint is the shearing of the rivets.

A riveted joint may also fail through a combination of the three preceding conditions.

There are other ways in which failure may occur, such as the shearing out of the portion of the plate *DHGE* (Fig. 40) in front of the rivet. But this danger can be taken care of by never placing the rivet center nearer the edge of the plate than 1½ diameters. Also, rivets should be spaced at least 3 diameters from center to center to allow room for the riveter. Figure 48 shows how a riveted plate may look after being loaded to failure.

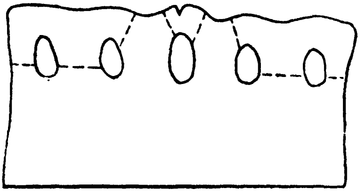


FIG. 48.

In determining the tensile stress on the net cross section, it is assumed that the stress is uniformly distributed. Experiment has shown that this assumption is not true but that there is a concentration of stress resulting in a distribution somewhat as shown in Fig. 49. Such a considerable departure from uniformity demands that a liberal factor of safety be employed.

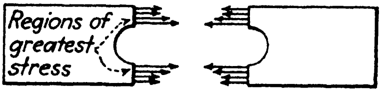


FIG. 49.

30. Notation.—The notation that has been used in the preceding articles on riveted joints and that will be used throughout the book is as follows:

s_t = stress in pounds per square inch in tension.
s_c = stress in pounds per square inch in compression or bearing.
s_s = stress in pounds per square inch in shear.
P_t = load that could be carried in tension by the net section.
P_c = load that could be carried in compression.
P_s = load that could be carried in shear.
P = load actually carried by a riveted joint.
P_g = load that could be carried in tension by the gross section.

31. Strength of Riveted Joints.—The "strength" of a riveted joint is understood to be the load it will safely carry in tension, bearing, shear, or combinations of these stresses

If the size of the plates to be used and the size, number, and manner of placing of the rivets are known, the strength of the joint is calculated for each of the stresses. The smallest of the results determines the load to be carried by the joint.

In the examples previously worked out, the rivet holes have been assumed to be of the same diameter as the rivets. In boiler, tank, and pipe joints the holes are usually drilled and are made $\frac{1}{16}$ in. larger than the cold rivet, so that the hot expanded rivet can be readily driven. In computing all stresses the diameter of the rivet hole is used. It is assumed that the rivet will fill the hole as a result of the expansion due to the driving. In structural work the rivet holes are usually punched; and although the hole will be $\frac{1}{16}$ in. larger than the cold rivet, it is customary to allow a diameter $\frac{1}{8}$ in. larger than the diameter of the cold rivet in computing the tensile stress in the plate, on the assumption that the plate is injured to this extent by the punching.

In the following article, several examples will be solved for the strength of typical joints.

Example 1.—Given a lap joint with 10- by $\frac{1}{2}$-in. plates, connected by four rivets, arranged in one row, in $\frac{7}{8}$-in. holes. Find the strength of the joint.

Solution.—The load that the joint will carry in tension is calculated from the section through the center of the row of rivets. The area of this section is $(10 - 4 \times \frac{7}{8}) \times \frac{1}{2} = 3.25$ sq. in.

If s_t is 11,000 lb. per sq. in., then the load is

$$3.25 \times 11{,}000 = 35{,}750 \text{ lb.}$$

Since there are four rivets in single shear, the hearing area is

$$4 \times \frac{\pi}{4}\left(\frac{7}{8}\right)^2 = 2.41 \text{ sq. in.}$$

If a safe stress in shear is 8,800 lb. per sq. in., then the load is

$$2.41 \times 8{,}800 = 21{,}210 \text{ lb.}$$

There are four rivets in bearing, the area for which is

$$4 \times \tfrac{7}{8} \times \tfrac{1}{2} = 1.75 \text{ sq. in.}$$

If a safe bearing stress is 19,000 lb. per sq. in., then the load is

$$1.75 \times 19{,}000 = 33{,}250 \text{ lb.}$$

Since the joint is weakest in shear, the strength of the joint is 21,210 lb.

Example 2.—Given a lap joint with 10- by ½-in. plates connected by eight ¾-in. rivets, arranged in two rows. Find the strength of the joint (Fig. 50). The rivet holes are ¾ in. in diameter.

Solution.—In analyzing this problem, the assumption is made that, of the entire load P which must be carried in tension by the cross section AB through the first row of rivets in the upper plate, one-half is transmitted to the lower plate by the rivets in the first row. The other half is carried by the section EF through the second row of rivets. Then the strength of the joint in tension is figured at the first row of rivets, because the tensile stress is higher than at the second section.

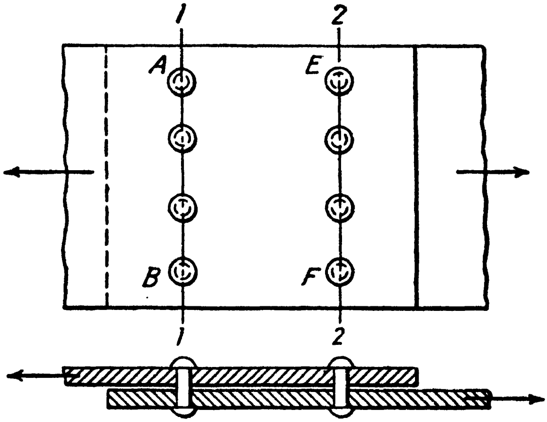


FIG. 50.

The effective area is the entire width minus the diameter of four rivet holes, times the thickness of the plate, or

$$A = (10 - 4 \times \tfrac{3}{4}) \times \tfrac{1}{2} = 3.5 \text{ sq. in.}$$

Since the safe tensile stress is 16,000 lb. per sq. in., the load in tension

$$P_t = 16{,}000 \times 3.5 = 56{,}000 \text{ lb.}$$

Each rivet is assumed to carry an equal part of the load in shear and bearing. The area of a ¾-in. rivet is 0.44 sq. in., and total shearing area

$$A = 8 \times 0.44 = 3.52 \text{ sq. in.}$$

Then the load P_s in shear, since 8,800 is a safe shearing stress, is

$$P_s = 8{,}800 \times 3.52 = 30{,}980 \text{ lb.}$$

There are eight rivets in bearing on which a safe stress of 18,000 lb. may be placed. From Eq. (7), the bearing area of a rivet is td. Then

$$A = 8 \times \frac{1}{2} \times \frac{3}{4} = 3 \text{ sq. in.}$$

The safe load P_c in bearing is

$$P_c = 18{,}000 \times 3 = 54{,}000 \text{ lb.}$$

Since the joint is weakest in shear, the load to be safely carried is 30,980 lb.

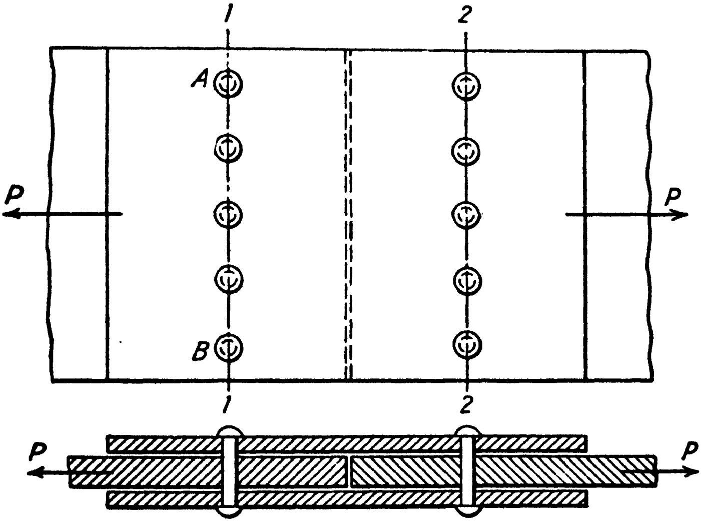


FIG. 51.

In butt joints, the plates placed above and below the main plates are usually called "butt straps" or "cover plates." They are selected so that the thickness of each is between $\frac{1}{2}t$ and t, where t is the thickness of the plate. As the assumption is made that one-half the load on the joint is carried by the butt strap from one plate to the other, if the thickness of the strap is more than one-half the main plate, the joint would fail by tension in the main plates before the straps gave way.

Example 3.—Two 12-in. plates $\frac{3}{4}$ in. thick are united by two $\frac{1}{2}$-in. cover plates to form a butt joint. The plates are connected by five rivets in single rows on each side of the joint. Find its strength (Fig. 51). The rivet holes are $\frac{7}{8}$ in. in diameter.

The strength of a butt joint is figured from the rivets on one side of the joint only, for the row of rivets AB must carry the entire load P through the cover plates from one plate to the other.

The smallest area of the main plate is the section AB through the row of rivets.

$$A = (12 - 5 \times \tfrac{7}{8})\tfrac{3}{4} = 5.72 \text{ sq. in.}$$

Since safe $s_t = 16{,}000$,

$$P_t = 16{,}000 \times 5.72 = 91{,}500 \text{ lb.}$$

There are five $\frac{7}{8}$-in. rivets resisting the shear, but each is in double shear. That is, there are ten shearing surfaces of area

$$A = 10 \times \frac{\pi}{4}\left(\frac{7}{8}\right)^2 = 10 \times 0.6 = 6 \text{ sq. in.}$$

If a safe shearing stress is $s_s = 8{,}800$ lb. per sq. in.,

$$\therefore P_s = 8{,}800 \times 6 = 52{,}800 \text{ lb.}$$

The bearing area of a rivet is $t \times d$. Then, for the five rivets,

$$A = 5 \times \tfrac{3}{4} \times \tfrac{7}{8} = \tfrac{105}{32} = 3.3 \text{ sq. in.}$$

For the safe bearing stress, $s_c = 18{,}000$ lb./sq. in.

$$\therefore P_c = 18{,}000 \times 3.3 = 59{,}400 \text{ lb.}$$

Since the joint is weakest in shear, the strength of the joint is 52,800 lb.

32. Pitch and Back Pitch.—When plates are united to form a riveted joint, the rivets in a line, as 1-1 or 2-2 in Fig. 52, are known as a *row*. The joint in this figure is said to have two rows of rivets. The distance between two adjacent rivets in the same row, such as AC or AB, is called the *pitch*. Each section AB or AC is of the same width and contains the same number of rivets. These sections are called *repeating sections*. The strength of a joint may be figured from the strength of a repeating section,

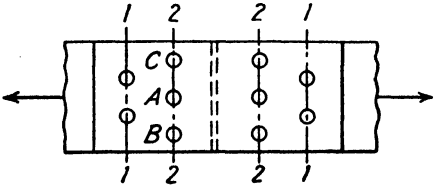


FIG. 52.

The distance between rows 1-1 and 2-2 is called the ***back pitch.***

The minimum distance between the centers of rivets shall be equal to approximately 3 diameters. The maximum pitch in a compression member shall not exceed 16 times the thickness of the thinnest outside plate, with a maximum of 12 in. The distance between rows, or the back pitch, shall not be less than 2 diameters. Usually the rivets are staggered, and the rule commonly followed is to make the diagonal distance between the rivets such a value that it will satisfy the equation

$$p' - d = 0.6(p - d) \tag{8}$$

where p' is the back pitch.

In accordance with this rule the back pitch would vary with the size of the rivets but would be close to $2\frac{1}{4}$ times the diameter of a rivet.

In boiler work the joint must be made steamtight. This condition, as well as the strength of the joint, controls the pitch. Handbooks and specifications should be consulted for additional detailed information.

Example 1.—Find the pitch p of a single-riveted lap joint, such that the tensile strength of the plate shall equal the shearing strength of the rivets, if $t = \frac{1}{2}$ and $d = \frac{3}{4}$. Use $s_t = 16{,}000$; $s_s = 8{,}800$; and $s_c = 18{,}000$.

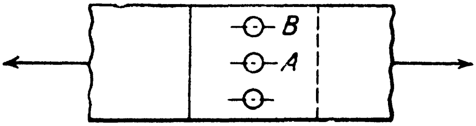


FIG. 53.

Since the joint is single-riveted, there will be but one rivet in each repeating section AB (Fig. 53).

Let $s_s = 8{,}800$ lb. Each rivet is in single shear.

$$\therefore P_s = \frac{\pi}{4} \times \left(\frac{3}{4}\right)^2 8{,}800 = 3{,}872 \text{ lb.}$$

Also, the tensile strength of a section of width p is

$$P_t = (p - d)\tfrac{1}{2} \times 16{,}000$$

But

$$(p - \tfrac{3}{4})\tfrac{1}{2} \times 16{,}000 = 3{,}872 \text{ lb.}$$
$$p = 1.23 \text{ in.}$$

For the bearing strength,

$$P_c = dt \times 18{,}000 = \tfrac{3}{4} \times \tfrac{1}{2} \times 18{,}000 = 6{,}750 \text{ lb.}$$

Then, if the rivets are placed 1.23 in. from center to center, the joint will be as strong in tension as in shear. The total load P that such a joint would carry is found by multiplying 3,872 lb., the strength of one section, by the number of repeating sections.

The total width of plate necessary would be 1.23 in. times the number of sections plus enough metal so that the centers of outside rivets shall not be nearer the edge of the plate than 1½ diameters.

Although this joint would be strong enough with a pitch of 1.23 in., the rivets should not be spaced nearer than 3 diameters or

$$3 \times \tfrac{3}{4} = 2.25 \text{ in.}$$

The bearing strength should be figured for safety.

Example 2.—Find the pitch of a double-riveted two-strap butt joint, for which $t = \frac{1}{2}$, $d = \frac{7}{8}$ in., such that the tensile stress shall be equal to the bearing stress.

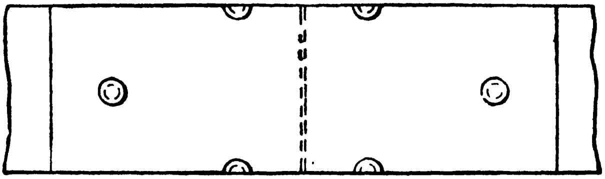


Fig. 54.

Solution.—The bearing stress is calculated first, for it is the one definitely known. The repeating section of this joint is shown in Fig. 54.

$$s_c = 18{,}000 \qquad t = \tfrac{1}{2} \qquad d = \tfrac{7}{8} \text{ in.} \qquad n = 2$$
$$\therefore P_c = \tfrac{1}{2} \times \tfrac{7}{8} \times 18{,}000 \times 2 = 15{,}750$$

Also,

$$P_t = (p - \tfrac{7}{8})\tfrac{1}{2} \times 16{,}000$$

But

$$P_c = P_t$$
$$\therefore (p - \tfrac{7}{8})\tfrac{1}{2} \times 16{,}000 = 15{,}750$$
$$p = 1.97 + 0.875 = 2.845 \text{ in.} \quad \textit{Ans.}$$

For shearing strength, there are two rivets in double shear for each section.

$$\therefore P_s = 2 \times 2 \times \frac{\pi}{4}\left(\frac{7}{8}\right)^2 \times 8{,}800 = 21{,}160 \text{ lb.}$$

The load P that such a joint will carry is obtained by multiplying the safe load on any repeating section by the number of such sections in the joint.

The shearing strength should be figured for safety.

Example 3.—Each repeating section of pitch 2⅞ in. of a double-riveted lap joint with ½-in. plates is to carry a load of 7,000 lb. Find the size of the rivets necessary to carry the load safely.

Solution.—The bearing and shearing strength must be at least equal to the tensile strength. Then,

$$P_c = 2 \times \tfrac{1}{2} \times d \times 18{,}000 = 7{,}000 \text{ lb.}$$

$$d = \frac{7{,}000}{18{,}000} = \frac{7}{18} = 0.4 \text{ in.}$$

$$P_s = \frac{2\pi d^2}{4} \times 8{,}800 = 7{,}000$$

$$\therefore d^2 = 0.506$$

$$d = 0.71$$

The larger diameter, 0.71, must be used. The nearest commercial size would be the one to choose.

Example 4.—Find the number of ¾-in. rivets necessary to carry a load of 12,000 lb. on each repeating section of a butt-strap joint, with main plates ¾ in. thick.

Solution.—As in the preceding example, we equate the tensile strength to each of the bearing and shearing strengths.

Then

$$P_c = n \times t \times d \times 18{,}000 = n \times \tfrac{3}{4} \times \tfrac{3}{4} \times 18{,}000 = 12{,}000$$

$$n = 1.2 \text{ rivets}$$

Since the rivets are in double shear,

$$P_s = 2n \times \frac{\pi}{4}\left(\frac{3}{4}\right)^2 \times 8{,}800 = 12{,}000$$

$$n = 1.55 \text{ rivets}$$

Then four rivets are required, two on each side of the joint.

33. Ultimate and Working Stresses.—The values of the ultimate stresses given by the Boiler Code of the American Society of Mechanical Engineers are as follows:

$$s_t = 55{,}000 \text{ lb./sq. in.}$$
$$s_c = 95{,}000 \text{ lb./sq. in.}$$
$$s_s = 44{,}000 \text{ lb./sq. in.}$$

If a factor of safety of 5 is used, then the safe working stresses are

$$s_t = 11{,}000 \text{ lb./sq. in.}$$
$$s_c = 19{,}000 \text{ lb./sq. in.}$$
$$s_s = 8{,}800 \text{ lb./sq. in.}$$

If the efficiency of a riveted joint is involved in a problem, either set of values for the stresses may be used. If a factor of safety other than 5 is specified, then the ultimate stresses must be used. Also, if the factor of safety is to be determined, the solution must be based on the ultimate stresses. In the problems that follow, unless otherwise stated the preceding unit stress values will be used exclusively.

34. Efficiencies, Net and Gross Area.—The area of the cross section of a plate is called the *gross area* or bt.

The net area is $(b - nd)t$, where n is the number of rivets in a section.

The quotient obtained by dividing the tensile strength of the gross area into the strength of the riveted joint is called the *efficiency* of the joint and is represented by E.

The strength of the joint is the smallest of P_t, P_s, and P_c. That is, the efficiency E is

$$E = \frac{P_t}{P_g} = \frac{(b - d)ts_t}{bts_t} = \frac{(b - d)t}{bt} = \frac{\text{net area}}{\text{gross area}} \tag{9}$$

or

$$E = \frac{P_s}{P_g} = \frac{\pi d^2 s_s}{4\,bts_t} \tag{10}$$

or

$$E = \frac{P_c}{P_g} = \frac{tds_c}{bts_t}. \tag{11}$$

The smallest of the three results expressed in percentage gives the *efficiency of the joint.* In determining the efficiency, the pitch width of plate may be used instead of the entire width, and d will represent the diameter of the hole as well as the diameter of the driven rivet.

Example 1.—Find the efficiency of a single-riveted lap joint, if $t = \frac{1}{2}$ in., $d = \frac{7}{8}$ in., and $p = 2$ in.

Solution.—$P_t = (2 - \frac{7}{8})\frac{1}{2} \times 11{,}000 = 6{,}190$ lb.

$$P_s = \frac{\pi}{4}\left(\frac{7}{8}\right)^2 \times 8{,}800 = 5{,}280 \text{ lb.}$$
$$P_c = \tfrac{1}{2} \times \tfrac{7}{8} \times 19{,}000 = 8{,}310 \text{ lb.}$$
$$P_u = 2 \times \tfrac{1}{2} \times 11{,}000 = 11{,}000 \text{ lb.}$$

Since P_s is the smallest, it is the strength of the joint, and, from Eq. (10),

$$E = \frac{5{,}280}{11{,}000} = 0.48 = 48 \text{ per cent}$$

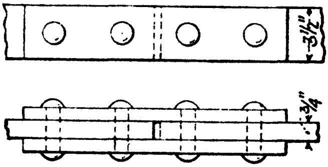


FIG. 55.

Example 2.—Find the efficiency of a double-riveted two-strap butt joint, if for the main plates $t = \frac{3}{4}$ in., $d = \frac{7}{8}$ in., and $p = 3.5$ in. (Fig. 55).

Solution.—The figure shows a pitch section of the joint. There are two rivets in bearing and two in double shear.

$$P_t = (3.5 - \tfrac{7}{8})\tfrac{3}{4} \times 11{,}000 = 21{,}660$$
$$P_s = 2 \times 2 \times 0.6 \times 8{,}800 = 21{,}120$$
$$P_c = 2 \times \tfrac{3}{4} \times \tfrac{7}{8} \times 19{,}000 = 24{,}940$$

For the unpunched plate,

$$P_u = 3.5 \times \tfrac{3}{4} \times 11{,}000 = 28{,}880$$

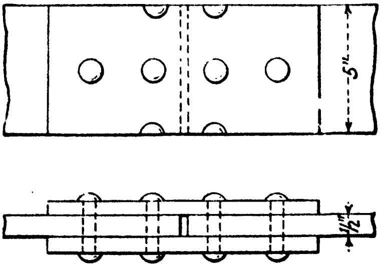


FIG. 56.

Since P_s is the strength of the joint,

$$E = \frac{21{,}120}{28{,}880} = 0.731 = 73.1 \text{ per cent}$$

Example 3.—A double-riveted butt joint with ½-in. main plates and ¾-in. rivets has a pitch of 2½ in. for the inner row and of 5 in. for the outer row. Compute its efficiency (Fig. 56).

Solution.—As a matter of illustration, the ultimate stress values will be used in the solution of this problem.

For a pitch section 5 in. wide,

$$\text{Ultimate } P_t \text{ (outer row)} = (5 - \tfrac{7}{8})\tfrac{1}{2} \times 55{,}000 = 113{,}400 \text{ lb.}$$
$$\text{Ultimate } P_t \text{ (inner row)} = (5 - 2 \times \tfrac{7}{8})\tfrac{1}{2} \times 55{,}000 = 89{,}380 \text{ lb.}$$

Each rivet is in double shear.

$$\text{Ultimate } P_s \text{ (one rivet)} = 2 \times 0.6 \times 44{,}000 = 52{,}800 \text{ lb.}$$

Since there are three rivets in shear,

$$\text{Ultimate } P_s \text{ for the joint} = 3 \times 52{,}800 = 158{,}400 \text{ lb.}$$
$$\text{Ultimate } P_c \text{ (one rivet)} = \tfrac{7}{8} \times \tfrac{1}{2} \times 95{,}000 = 41{,}560 \text{ lb.}$$
$$\text{Ultimate } P_c \text{ for the joint} = 3 \times 41{,}560 = 124{;}680 \text{ lb.}$$

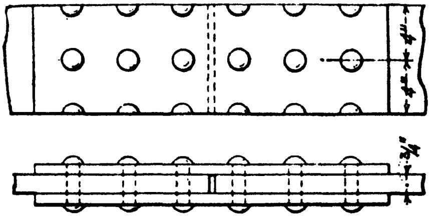


FIG. 57.

The joint might fail by tearing along the second row and the rivet in the front row failing in shear, or in bearing. These strengths are

Tearing second row + shear of first rivet =
$$89{,}380 + 52{,}800 = 142{,}180 \text{ lb.}$$
Tearing second row + bearing on front rivet =
$$89{,}380 + 41{,}560 = 130{,}940 \text{ lb.}$$

A comparison shows $P_t = 113{,}400$ lb. is the smallest of the values, and therefore is the ultimate strength.

The gross strength = 5 × ½ × 55,000 = 137,500 lb.

$$\therefore E = \frac{113{,}400}{137{,}500} = 0.825 = 82.5 \text{ per cent}$$

In double-riveted joints, it is seldom necessary to investigate failure from the combination of stresses. But in triple- and quadruple-riveted joints, such ways of failure must be considered if the pitch is different in each of the rows.

Example 4.—A triple-riveted butt joint has ¾-in. main plates, $\frac{7}{16}$-in. cover plates, 1-in. rivets, and a pitch of 4 in. in all rows. Find the efficiency (Fig. 57).

Solution.

$$P_t = (4 - 1)\tfrac{3}{4} \times 11{,}000 = 24{,}750 \text{ lb.}$$
$$P_s = 3 \times 2 \times 0.7854 \times 8{,}800 = 41{,}470 \text{ lb.}$$
$$P_c = 3 \times 1 \times \tfrac{3}{4} \times 19{,}000 = 42{,}750 \text{ lb.}$$
$$P = 4 \times \tfrac{3}{4} \times 11{,}000 = 33{,}000 \text{ lb.}$$
$$\therefore E = \frac{P_t}{P} = \frac{24{,}750}{33{,}000} = 0.75 = 75 \text{ per cent}$$

No attention need be paid to strength by combined stresses, for the tensile strength P_t of the sections through each row of rivets is the same. The results show that the efficiencies in shear and bearing are much higher than in tension. They may be brought more nearly equal by using smaller rivets or thicker plates or by making the pitch of the outer row double the second row and four times that of the inner rows. The last method will be illustrated in the next example.

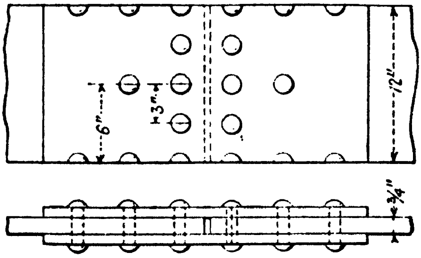


Fig. 58.

Example 5.—In Example 4, let the pitch be 3 in. for the inner row, 6 in. for the second, and 12 in. for the outer row. Find the efficiency of the joint (Fig. 58).

Solution.—For the repeating section, p = 12 in.

$$P_t \text{ (outer row)} = (12 - 1)\tfrac{3}{4} \times 11{,}000 = 90{,}750 \text{ lb.}$$
$$P_t \text{ (middle row)} = (12 - 2)\tfrac{3}{4} \times 11{,}000 = 82{,}500 \text{ lb.}$$
$$P_t \text{ (inner row)} = (12 - 4)\tfrac{3}{4} \times 11{,}000 = 66{,}000 \text{ lb.}$$

Shearing strength for one rivet = 2 × 0.7854 × 8,800 = 13,800 lb.

$$P_s = 7 \times 13{,}800 = 96{,}600 \text{ lb.}$$

Bearing strength of one rivet = 1 × ¾ × 19,000 = 14,250 lb.

$$P_c = 7 \times 14{,}250 = 99{,}750 \text{ lb.}$$

Tension (second row) + shear = 82,500 + 13,800 = 96,300 lb.
Tension (inner row) + shear = 66,000 + 3 × 13,800 = 107,400
There are three rivets in shear in front of the inner row of rivets.

Since the bearing strength of one rivet is greater than its shearing strength, we do not consider bearing in combination with tension.

$$P_t = 90{,}750 \text{ is the least}$$

$$\therefore E = \frac{90{,}750}{12 \times \frac{3}{4} \times 11{,}000} = 0.917 = 91.7 \text{ per cent}$$

$$\text{Efficiency in shear} = \frac{96{,}600}{99{,}000} = 97 \text{ per cent}$$

$$\text{Efficiency in bearing} = \frac{99.750}{99{,}000} = 100.8 \text{ per cent}$$

These results show a more economical joint than the one in Example 4. An efficiency of over 100 per cent in bearing as in the previous problem means that the joint is stronger in bearing than the main plate in tension and that therefore a smaller sized rivet might have been used.

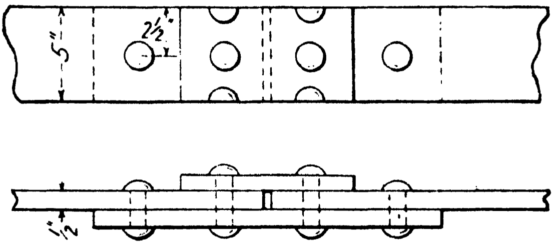


Fig. 59.

Example 6.—A double-riveted butt joint, with ½-in. main plates, $\frac{5}{16}$-in. cover plates, and ¾-in. rivets, has one cover plate wider than the others. The pitch of the outer row is 5 in., and that of the inner 2½ in. Find the efficiency of the joint (Fig. 59).

Solution.

$$P_t \text{ (outer row)} = (5 - \tfrac{3}{4})\tfrac{1}{2} \times 11{,}000 = 23{,}380 \text{ lb.}$$
$$P_t \text{ (inner row)} = (5 - 2 \times \tfrac{3}{4})\tfrac{1}{2} \times 11{,}000 = 19{,}250 \text{ lb.}$$

Shearing strength of one rivet section = 0.442 × 8,800 = 3,890 lb. Since there are two rivets in double shear and one in single shear, in each pitch section,

$$P_s = 5 \times 3{,}890 = 19{,}450 \text{ lb.}$$

$$\text{Bearing strength of one rivet} = \tfrac{1}{2} \times \tfrac{3}{4} \times 19{,}000 = 7{,}125 \text{ lb.}$$

$$P = 3 \times 7{,}125 = 21{,}375 \text{ lb.}$$

For tension in inner row with shear of the outer rivet the strength is

$$\therefore 19{,}250 + 3{,}890 = 23{,}140 \text{ lb.}$$

Since the bearing strength of a rivet is greater than the shearing strength, it need not be considered with tension.

For shear in the outer rivet, with the bearing in the two inner rivets the strength is

$$3{,}890 + 2 \times 7{,}125 = 18{,}140 \text{ lb.}$$

Since the shearing strength of the inner rivets is greater than bearing, the combination of shear in inner rivets with bearing in outer need not be considered.

Then the strength of the joint is 18,140.

$$P \text{ for the gross area} = 5 \times \tfrac{1}{2} \times 11{,}000 = 27{,}500 \text{ lb.}$$

$$\therefore E = \frac{18{,}140}{27{,}500} = 65.9 \text{ per cent } \textit{Ans.}$$

35. Riveted Joints of Maximum Efficiency.—Where the type of joint has been decided upon, it has a maximum efficiency when all three strengths are equal. Since all three strengths are to be equal, the efficiency of the joint is

$$E = \frac{P_t}{P_u} = \frac{(p - d)ts_t}{pts_t} = \frac{p - d}{p} \tag{12}$$

The maximum efficiency may always be obtained by dividing the net width of plate by the gross width.

Structural joints are usually not joints of maximum efficiency. The problem is very largely one of finding the number of rivets required to make the joints secure for both shearing and bearing. The width and the thickness of the members to be joined are generally determined from considerations other than those relating purely to the joint.

36. Derivations.—Some general relations between the various parts of the joint, when maximum efficiency is obtained, may be derived.

Since

$$P_t = P_c = P_s \text{ for maximum efficiency}$$

and

$$P_t = (p - d)ts_t; \qquad P_s = n\frac{\pi d^2}{4}s_s; \qquad P_c = mdts_c$$

where n is the number of shear areas, m is the number of compression surfaces, and the remaining symbols are as listed under Notation, Art. 30.
Then

$$(p - d)ts_t = \frac{n\pi d^2}{4}s_s = mdts_c$$

For a single-riveted lap joint, $n = 1$, and $m = 1$. Equating between shear and compression, we have

$$\frac{\pi d^2}{4}s_s = dts_c$$

Solving for d in terms of t produces

$$d = \frac{4s_c}{\pi s_s}t = \frac{4 \times 19{,}000}{\pi \times 8{,}800}t = 2.75t$$

Also,

$$(p - d)ts_t = dts_c$$

Solving, we have

$$p - d = \frac{s_c d}{s_t} = \frac{19{,}000}{11{,}000}d = 1.73d$$
$$p = 1.73d + d = 2.73d$$
$$p = 2.73 \times 2.75t = 7.5t$$

Thus, in a single-riveted lap joint, $p = 7.5t$, and $d = 2.75t$ for maximum efficiency, which is

$$E = \frac{p - d}{p} = \frac{7.5t - 2.75t}{7.5t} = \frac{4.75}{7.5} = 0.633 = 63.3 \text{ per cent}$$

Since the commercial sizes must be used, it is not always possible to apply the foregoing relations rigidly.

For a single-riveted butt joint

$$(p - d)ts_t = 2\frac{\pi d^2}{4}s_s = dts_c$$

Then, from $P_s = P_c$,

$$d = \frac{2 \times 19{,}000}{\pi \times 8{,}800}t = 1.375t$$

And from $P_t = P_c$,

$$p = \frac{19{,}000}{11{,}000} \times d + d = 2.73d$$

Also,

$$p = 2.73 \times 1.375t = 3.75t$$

Thus, in a single-riveted butt joint, $p = 3.75t$, $d = 1.375t$ for maximum efficiency, which is

$$E = \frac{p - d}{p} = \frac{7.5t - 2.75t}{7.5t} = \frac{4.75}{7.5} = 0.633 = 63.3 \text{ per cent}$$

The efficiencies of the single-riveted lap and butt joints prove to be the same.

For the double-riveted lap joint,

$$(p - d)ts_t = 2\frac{\pi d^2}{4}s_s = 2dts_c$$

From $P_s = P_c$,

$$d = \frac{4 \times 19{,}000}{\pi \times 8{,}800}t = 2.75t$$

From $P_t = P_c$,

$$p = \frac{2 \times 19{,}000}{11{,}000}d + d = 4.45d$$

Also,

$$p = 4.45 \times 2.75t = 12.25t$$

Thus, in a double-riveted lap joint, $p = 12.25t$, $d = 2.75t$ for a maximum efficiency, which is

$$E = \frac{p - d}{p} = \frac{12.25t - 2.75t}{12.25t} = 0.776 = 77.6 \text{ per cent}$$

In a similar manner, the proportions of the various parts of the joint and its efficiency may be worked out for other types.

For the double-riveted butt joint,

$$(p - d)ts = \frac{4\pi d^2}{4}s_s = 2dts_c$$

From $P_s = P_c$,

$$d = \frac{2t \times 19{,}000}{\pi \times 8{,}800} = 1.375t$$

From $P_t = P_c$,

$$p = \frac{2 \times 19{,}000d}{11{,}000} + d = 4.45d$$

Also,

$$p = 4.45 \times 1.375t = 6.125t$$

Thus in a double-riveted butt joint, $p = 6.125t$, $d = 1.375t$ for a maximum efficiency, which is

$$E = \frac{p - d}{p} = \frac{6.125t - 1.375t}{6.125t} = \frac{4.75}{6.125} = 77.6 \text{ per cent.}$$

Example 1.—Design a single-riveted lap joint of maximum efficiency with ½-in. plates.

Solution.—$P_t = P_s$, $P_t = P_c$, and $P_s = P_c$.

$$P_t = (p - d)\tfrac{1}{2} \times 11{,}000$$

$$P_s = \frac{\pi}{4}d^2 \times 8{,}800$$

and

$$P_c = \tfrac{1}{2} \times d \times 19{,}000$$

First equating P_s and P_c, we have

$$\frac{\pi}{4}d^2 8{,}800 = \frac{1}{2}d 19{,}000$$

$$d = \frac{19{,}000 \times 4}{2\pi \times 8{,}800} = 1.374 \text{ in.} = 1\tfrac{3}{8} \text{ in.}$$

This diameter is larger than any commercial size, and such a design is impossible for a boiler joint. A single-riveted butt joint could be used. (As an exercise, verify this statement.)

Example 2.—Design a double-riveted butt joint of maximum efficiency with ⅞-in. rivets.

Solution.

$$P_t = (p - \tfrac{7}{8})t \times 11{,}000$$

$$P_c = 2 \times \tfrac{7}{8} \times t \times 19{,}000$$

$$P_s = 4 \times \frac{\pi}{4}\left(\frac{7}{8}\right)^2 \times 8{,}800$$

For maximum efficiency,

$$(p - \tfrac{7}{8})t \times 11{,}000 = 2 \times \tfrac{7}{8} \times t \times 19{,}000$$
$$p = \tfrac{7}{4} \times \tfrac{19}{11} + \tfrac{7}{8} = 3.895$$

Also,

$$2 \times \tfrac{7}{8} \times t \times 19{,}000 = 4 \times 0.601 \times 8{,}800$$
$$t = 0.64 \text{ in.}$$
$$E = \frac{p - d}{p} = \frac{3.895 - 0.875}{3.895} = 77.9 \text{ per cent}$$

The nearest commercial sizes should be chosen.

Example 3.—For a double-riveted lap joint, find d and p in terms of t for maximum E.

$d = 2.75t$; $p = 12.25t$; $E = 77.5$ per cent *Ans.*

Example 4.—Solve Example 3 for a triple-riveted lap joint having the same pitch in all rows.

$d = 2.75t$; $p = 17t$; $E = 83.8$ per cent *Ans.*

Example 5.—Solve example 3 for a single-riveted butt joint.

$d = 1.37t$; $p = 3.75t$; $E = 63.3$ per cent *Ans.*

Example 6.—Solve Example 3 for a double-riveted butt joint, with the pitch of the outer row double that of the inner.

$d = 1.37t$; $p = 8.5t$; $E = 83.8$ per cent *Ans.*

37. Practical Considerations.—The riveted joint of maximum efficiency is one in which the least amount of material is used. But economy is not always the chief requisite. In riveted boilers and tanks, riveted joints must be steam- and watertight. For designing such joints, the following empirical relations are widely used:

For lap joints,

$$d = 1.5\sqrt{t} \tag{13}$$

For butt joints,

$$d = 1.3\sqrt{t} \tag{14}$$

If neither t nor d is given but the type of joint is known, an efficiency is assumed not greater than the maximum efficiency for this type of joint. The value for d may then be found from the empirical equations. If now the strength in tension is equated to the strength in bearing, and the strength in tension to the strength in shear, two values for the pitch will be found. The smaller of the two is the one to use in the design of the joint.

Example 1.—Design a double-riveted lap joint for a hot-water tank, 18 in. in diameter, with a 100 lb. per sq. in. pressure.

Solution.—See Example 3, Art. 36 for maximum E.

$$2s_t tE = Dp$$
$$2 \times 11{,}000t \times 0.775 = 18 \times 100$$
$$t = 0.106$$
$$d = 1.5\sqrt{t} = 1.5\sqrt{0.106} = 0.486$$

Use $t = \frac{1}{8}$ in.; $d = \frac{1}{2}$ in.
Now, equating P_t to P_s and P_c, respectively,

$$\left(p - \frac{1}{2}\right)\frac{1}{8} \times 11{,}000 = \frac{2\pi}{4}\left(\frac{1}{2}\right)^2 8{,}800$$
$$p = 3.01 \text{ in.}$$

Also,

$$(p - \tfrac{1}{2})\tfrac{1}{8} \times 11{,}000 = 2 \times \tfrac{1}{2} \times \tfrac{1}{8} \times 19{,}000$$
$$p = 2.23 \text{ in.}$$

The last value, being the smaller, is the one to use.
The actual efficiency is

$$E = \frac{p - d}{p} = \frac{2.23 - 0.5}{2.23} = 77.5 \text{ per cent}$$

which in this case is the maximum.

Example 2.—Design a single-riveted lap joint for a steel water pipe 42 in. in diameter, under a pressure of 60 lb. per sq. in.

Solution.—For a single-riveted lap joint,

$$\text{Maximum } E = 63.3 \text{ per cent}$$

Then

$$2 \times 11{,}000t \times 0.633 = 42 \times 60$$
$$t = 0.18 \text{ in.}$$

But

$$d = 1.5\sqrt{t} = 1.5\sqrt{0.18} = 0.636 \text{ in.}$$

Use $t = \frac{3}{16}$ in. and $d = \frac{3}{4}$ in.

Equating P_t to P_c and P_s, respectively,

$$\left(p - \tfrac{3}{4}\right)\tfrac{3}{16} \times 11{,}000 = \tfrac{3}{4} \times \tfrac{3}{16} \times 19{,}000$$
$$p = 2.05 \text{ in.}$$
$$\left(p - \frac{3}{4}\right)\frac{3}{16} \times 11{,}000 = \frac{\pi}{4}\left(\frac{3}{4}\right)^2 \times 8{,}800$$
$$p = 2.62 \text{ in.}$$

Therefore, use $p = 2.05$ in.

$$E = \frac{2.05 - 0.75}{2.05} = 0.633 = 63.3 \text{ per cent}$$

38. Stress Relations.—In the construction of riveted joints, the rivet holes are usually punched. Sometimes the plate needed is so thick that it cannot be punched without exceeding the strength of the punch, and in this case the holes must be drilled. If the diameter of the rivet hole is small, then the force that must be applied to the punch will produce an excessive compressive stress and the punch will break. A relation may be derived between the diameter of the hole and the thickness of the plate. Let s_c be the ultimate compressive strength of the punch and s_s the ultimate shearing strength of the plate, t the thickness of the plate, and d the diameter of the punch. The compressive force applied to the punch must be equal to the total shearing resistance of the plate. This may be expressed as follows:

$$\frac{\pi d^2}{4} s_c = \pi d t s_s$$
$$d = \frac{4 s_s t}{s_c}$$

If we know the ultimate strengths of the punch and plate, we can compute whether or not it is possible to punch the hole without breaking the punch. For example, let us assume that s_s is 44,000 lb. per sq. in., s_c, 120,000 lb. per sq. in., and t, $\frac{1}{2}$ in.; then the smallest hole that can be punched is

$$d = \frac{4 \times 44{,}000 \times 0.5}{120.000} = 0.73 \text{ in.}$$

What would be the stress in the punch if we wish to make a ½-in. hole? Then

$$s_c = \frac{4 \times 44{,}000 \times \frac{1}{2}}{\frac{1}{2}} = 176{,}000 \text{ lb./sq. in.}$$

and the ½-in. hole cannot be punched.

39. Working Stresses for Structural Joints.—The Special Committee of the American Society of Civil Engineers on Specifications for Bridge Design recommends the following safe stresses:

Kind of Stress	Pounds per Square Inch
Tension	18,000
Shear:	
Shop-driven rivets	12,000
Power-driven rivets	10,000
Hand-driven rivets	7,500
Bearing:	
Shop-driven rivets	24,000
Power-driven rivets	20,000
Hand-driven rivets	15,000

According to the American Institute of Steel Construction code, the following values are given for hand-driven rivets.

Kind of Stress	Pounds per Square Inch
Tension	16,000
Shear	10,000
Bearing	20,000

In the problems that follow, all that are purely structural may be solved by either of the foregoing sets of stresses. Several designs may be worked out for each problem by assuming that the rivets are shop-, power-, or hand-driven.

40. Welding.—For many purposes, welded connections are found to be more suitable than riveted connections. Welding is a process of joining two pieces of metal by fusion. The pieces to be joined are heated, and molten material of the same composition as the pieces is deposited between them. The heating is accomplished by means of an electric arc or an oxyacetylene torch. Lap and butt joints are made in various combinations. Figure 60 shows what is known as a "side fillet" weld. The fused metal is deposited in the angle between the two pieces. Shearing stresses are developed in this weld when either tensile or compressive forces are applied to the long axis of the plates. Figure 61 shows an "end" weld. This type of weld is not used by itself, but rather as an auxiliary to the side welds. The end weld is subjected to both tensile and shearing stresses.

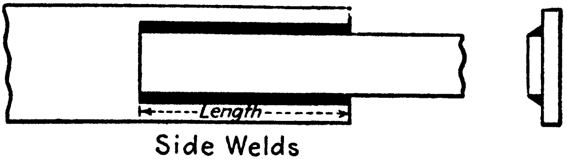


FIG. 60.

The use of welds in making connections is very general. A large number of metals can be welded successfully. Among these are, cast iron, cast steel, structural steel, tool steel, bronze, aluminum and copper.

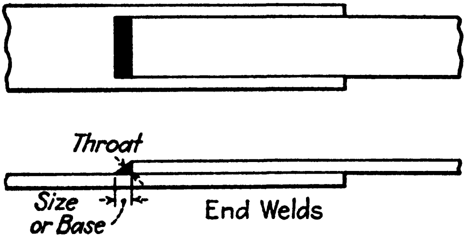
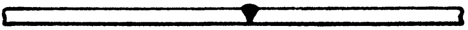


FIG. 61.

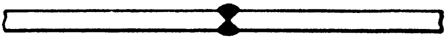
Lap welds are used to connect the members in roof trusses. Likewise, in the construction of a floor system, the beams are joined to each other and to the columns by means of lap welds. Another general use is in tank and boiler work.

Butt-welded joints are formed by butting the plates together and welding the edges. Up to $\frac{1}{2}$ in. in thickness the plates are beveled on one side to form a V which is then fused and filled and known as a "V" weld (Fig. 62). In a joint using plates more than ½ in. in thickness, the plates are beveled on both sides to form a double V, and the joint is known as a "double-V" weld (Fig. 63). The double-V weld ensures a more satisfactory joint than is the case when all the metal is added from one side. In all welded joints, there may be some change in the properties of the metals due to the heating, and it may be necessary to anneal the joint. This annealing will tend to remove the localized stresses that have been set up. Butt welds have a large variety of uses, of which, one of the most important is in repairing breaks. For new work, the butt weld is used in constructing steel pipe lines for gas, steam, or oil. It is also used in a variety of open as well as pressure tanks.



Single V-weld

FIG. 62



Double V-weld

FIG. 63.

The following resistances per linear inch are commonly used for side and end welds. The dimensions refer to the bases of the welds.

Base, Inches	Resistance, Pounds per Linear Inch
¼	2,000
⅜	3,000
½	4,000

Values of other dimensions of the base of the weld are proportional to these. In boiler construction a high-grade double-V butt weld is assumed to have a strength of about 90 per cent of the original plate and an average single-V weld about 70 per cent.

In butt welds the joint is subjected to a tension through the throat of the weld. The allowable tension is 13,000 lb. per sq. in.

Example 1.—A lap-welded joint is to resist a pull of 50,000 lb. The plates are ½ in. in thickness. What must be the length of the two side welds?

Solution.—A side weld for ½-in. plate will require a ½-in. fillet weld. From the table the resistance is seen to be 4,000 lb. per lin. in. Then

$$\text{Length} = \frac{50{,}000}{4{,}000} = 12.5 \text{ in.}$$

Since this length is applied to two fillets, there will be a length of 12.5/2 = 6.25 in. for each side weld.

Example 2.—A steel pipe 4 ft. in diameter and made of ⅜-in. metal has a longitudinal butt joint. What safe internal pressure could the joint withstand?

Solution.—From Eq. (4),

$$p = \frac{st}{r} = \frac{13{,}000 \times \frac{3}{8}}{24} = 203.1 \text{ lb./sq. in.}$$

The answer is worked out in terms of the strength of the weld, but it should be noted that a weld properly made is usually stronger than the plate itself.

Problems

37. In a side weld shown in Fig. 60, the narrow plate is 6 by ⅜ in. and resists a pull of 16,000 lb. What must be the length of the two side welds?

38. In Prob. 37, could a single end weld resist the pull?

39. In a lap weld the narrow plate is 4 by ½ in. and is subjected to a pull of 20,000 lb. What must be the length of the two side welds if 1 in. of end weld is also to be used?

40. A steel pipe 2 ft. in diameter is constructed from ¼-in. plates. What is the allowable working pressure on the assumption that longitudinal butt-welded joints are used?

41. A cylindrical water tank is 40 ft. high and 25 ft. in diameter. The bottom plates are ½ in. thick. What will be the factor of safety of the longitudinal butt-welded joints?

42. A lap joint is made by connecting two 4- by ½-in. plates with two ¾-in. rivets. What safe load will the joint carry?

43. Two plates, each 6 in. wide and ½ in. thick, are connected by ¾-in. rivets to form a lap joint. How many rivets will be needed if the pull on the joint is 15,000 lb.?

44. In a lap joint connecting ¾-in. plates, how many ⅞-in. rivets will be needed to carry a pull of 25,000 lb.?

45. Two ½-in. plates are connected by three ⅞-in. rivets to form a lap joint. What pull will the joint safely carry? What must be the width of the plates if two of the rivets are in one row?

46. In a lap joint connecting ½-in. plates, how many ¾-in. rivets are needed to carry a pull of 20,000 lb?

47. Two 4- by ¾-in. plates are joined to form a double-riveted butt joint, using two ½-in. cover plates. There are two ⅞-in. rivets on each side. If the pull on the joint is 24,000 lb., what are the values of the various unit stresses?

48. A standpipe in a small town is 16 ft. in diameter. The plates near the bottom are ½-in. thick. What is the unit tensile stress in the plate near the bottom when there is 60 ft. of water in the tank? The longitudinal joints are double-riveted lap joints.

What should be the pitch of the rivets and the rivet diameter? If the standpipe alone weighs 76,000 lb. and the footing weighs 110,000 lb., what must be the size of a square footing if the allowable soil pressure is 5,000 lb. per sq. ft. (Fig. 64)?

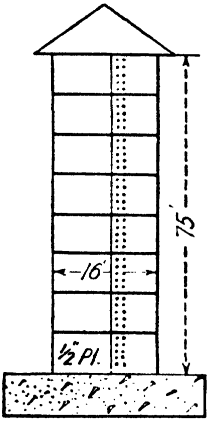


FIG. 64.

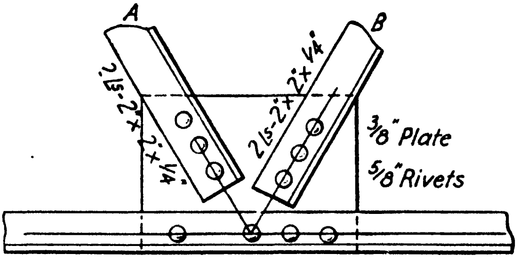


FIG. 65.

49. In a butt joint connecting ⅜-in. plates, how many ¾-in. rivets are needed to carry 80,000 lb. safely? What would be the number of rivets for a lap joint?

50. Find the efficiency of a double-riveted lap joint, if t is ½ in., d is ⅞ in., and p is 3 in.

51. The joint at a point in the lower chord of a roof truss is shown in Fig. 65. If the compression in member A is 18,000 lb. and the tension in member B is 14,000 lb., how many ⅝-in. rivets are needed to fasten these members to the ⅜-in. gusset plate? Rivet holes are ⅛ in. larger than rivets.

52. A boiler 6 ft. in diameter carries steam at a pressure of 120 lb. per sq. in. The plates are ⅜ in. thick. The rivets are ¾ in. in diameter, and the pitch is 3 in. If the longitudinal joint is a double-riveted lap, what are the several unit stresses? What is the efficiency?

53. A boiler 8 ft. in diameter is to carry a steam pressure of 160 lb. per sq. in. If the allowable unit stress in the plates is 10,000 lb. per sq. in. and the efficiency of the joint is taken as 80 per cent, what should be the thickness of the plates?

54. What force is required to punch a 1-in. hole in a ½-in. plate if the ultimate shearing strength of the plate is 44,000 lb. per sq. in.? What will be the compressive unit stress in the punch? If the original length of the hard-steel punch is 4 in., how much will it be shortened owing to the punching operation?

55. A water tank 40 ft. in diameter and 60 ft. high is built of ½-in. plates. It has double-riveted lap joints with 1-in. rivets and a 3-in. pitch. What is the factor of safety when the tank is full?

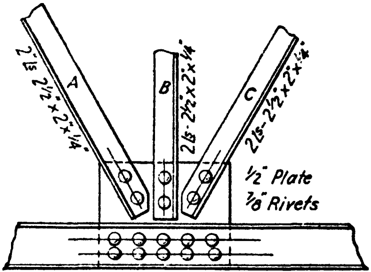


Fig. 66.

56. A gas tank is to be operated under a pressure of 60 lb. per sq. in. If the allowable tensile strength is 10,000 lb. per sq. in., what will be the thickness of the plate, on the assumption of a longitudinal double-riveted butt joint with an efficiency of 75 per cent?

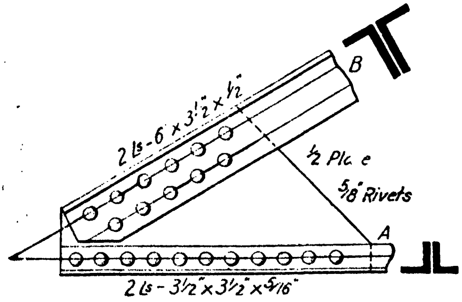


Fig. 67.

57. How many ⅞-in. rivets are needed to fasten the members *A*, *B*, and *C* to the ½-in. gusset plate (Fig. 66)? The stresses in the members *A*, *B*, and *C*, are, respectively, 18,000 lb. compression, 8,000 lb. compression, and 16,000 lb. tension. Assume the rivet holes 1 in. in diameter.

58. Figure 67 shows the lower chord of a roof truss. Eleven ¾-in. rivets are used to fasten the member *A* to the ⅜-in. gusset plate. The tensile stress in the member *A* is 56,000 lb. Is the riveting safe? What may be the compressive stress in member *B*, on the assumption that the rivets are used to their full capacity?

59. A riveted steel water pipe is 2 ft. in diameter and is made of ⅜-in. plates. The single-riveted longitudinal joint has ¾-in. rivets with a pitch of 2½ in. What is the factor of safety?

60. The dimensions of a single-riveted two-strap butt joint are as follows: $t = \frac{1}{2}$ in., $d = \frac{7}{8}$ in., $p = 3$ in. What is the efficiency of the joint?

61. Design a double-riveted butt joint for a tank 40 ft. in diameter and 60 ft. high with a factor of safety of 5.

62. Design a lap joint for steel plates to resist a pull of 16,000 lb.

63. A cylindrical oil tank is 100 ft. in diameter and 28 ft. high. Using ½-in. plates, work out a suitable butt joint. (Oil weighs 56 lb. per cu. ft.)

64. In Prob. 63, would a butt weld be strong enough?

65. In the boiler of Prob. 52, would a butt weld be suitable as a substitute for the lap joint?

66. A compressed-air tank is 36 in. in diameter. It is made of ¼-in. plates. If the strength of the longitudinal butt welds is assumed to be 85 per cent of the strength of the plate, what safe working pressure may be permitted? Use American Society of Mechanical Engineers values with a safety factor of 5.

67. In Fig. 68, what would be the value of the permissible reaction? The rivets are ⅞ in. in diameter. The web of the 10-in. I beam is 0.31 in. thick. The angles are 4 by 4 by ½ in. The flange of the 10-in. CB column section is ¾-in. thick.

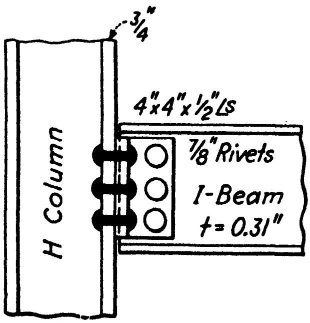


Fig. 68.

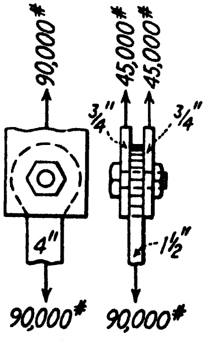


Fig. 69.

68. A steel eyebar carrying 90,000 lb. is shown in Fig. 69. What should be the diameter of a hard-steel pin? The allowable shearing stress is 12,000 lb. per sq. in., and the allowable bearing stress is 24,000 lb. per sq. in.

69. A clevis shown in Fig. 70 is used to hitch a road drag to a tractor. What safe pull could the tractor exert on the drag?

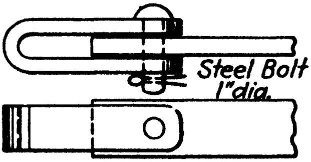


Fig. 70.

70. A cylindrical storage tank for compressed air has an inside diameter of 28 in. and a plate thickness of $\frac{3}{8}$ in. The longitudinal double-riveted lap joint has a pitch of 3 in. in each row. The ends are held by single-riveted lap joints, each joint having 30 rivets. All rivets are $\frac{7}{8}$ in. in diameter with $\frac{15}{16}$-in. rivet holes. What are the stresses in the joint when the air pressure in the tank is 150 lb. per sq. in.?

71. A spherical gas container is 36 ft. in diameter. The shell is made of $\frac{3}{4}$-in. steel plate. The great-circle joint is a double-riveted butt joint of maximum efficiency. If the gas pressure is 50 lb. per sq. in., what are the stresses?

CHAPTER III

MECHANICAL PROPERTIES OF MATERIALS

41. Basis of Design.—The designer of machinery and structures must know both the action of the forces that are applied and the strength of the various members working together to resist the forces successfully. It is essential for the members to be amply strong; on the other hand, any part that is unnecessarily strong means a waste of material. It is therefore highly important for the engineer to know the mechanical and physical properties of the timber, iron, steel, concrete, stone, etc., with which he works. In the solution of his problems, a knowledge of the laws of mechanics alone is insufficient and must be accompanied by a liberal knowledge of the properties of the materials. Much of this knowledge is gained from experiment and experience.

Properties of Materials.—The most important mechanical and physical properties are stiffness, elasticity, toughness, ductility, hardness, malleability, and strength.

The *stiffness* of a body is its ability to resist change in size and shape when a load is applied. In most building materials the stiffness is proportional to the load placed on the body.

The *strength* of a material depends on the load it will carry in tension, compression, and shear. The strength is ordinarily determined by carefully placing on the body steadily increasing loads. The load that a beam will carry, for example, can be altered very materially by suddenly applied impact loads or by the repeated application and removal of the load. These special cases require separate consideration and are beyond the scope of this book.

The *elasticity* of materials is the property by means of which they return to their original shape and size on the removal of the load. Any body when subjected to the action of forces undergoes changes in size and shape, depending on the amount of force and the material from which the body is made. The elastic property of materials varies to a large extent. Rubber, for example, may be stretched to several times its original length without permanent injury. A material such as putty when deformed remains so. Thus, it is said to be *plastic.* Most building materials are elastic up to a certain limit. Beyond that point, they only partly recover their original shape and size. Beyond the point of elasticity, such materials are said to be *partly plastic.*

A material that may be twisted, bent, or stretched until it is greatly deformed before it ruptures under a high stress is said to possess *toughness.* Rolled steel is an example of a tough material. Toughness is a very desirable property for materials in structures and machines that must withstand shocks without changes in shape.

When a material may be drawn out or elongated to an appreciable extent before rupture occurs, it is said to be *ductile* or to possess *ductility.*

Hardness is that quality of a material which tends to make it resistant to wear. This quality is of special importance in the construction of bearings or, in general, of wearing surfaces.

When a material may be hammered into desired shapes, it is *malleable.*

42. Alloys.—An alloy is a mixture of two or more metals. The utility of the alloy arises from the fact that pure metals are too soft or too weak to be used alone. Alloy steels owe their distinctive properties to various elements other than carbon. These alloy steels include vanadium steel, nickel steel, silicon steel, chrome vanadium steel, nickel chromium steel, etc. They take the name of the element having the greatest influence on their characteristics. Alloy steels are used where toughness and great strength are needed and have a wide application in machinery, automobiles, and construction.

43. Elastic Properties of Steel.—A specimen of steel of the shape shown in Fig. 71 is placed in a testing machine with the ends E and F in the jaws of the machine. The specimen is of such length that there will be a cylinder 8 in. long from A to B. The reasons for such lengths and shapes cannot be discussed here. It is sufficient to point out that, in order that the results of different tests may be compared, the specimens tested must be of the same size and shape.

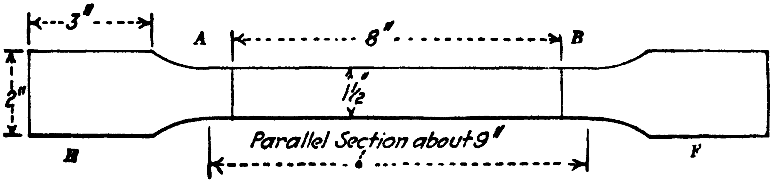


FIG. 71.

A load is now applied by the machine, putting the bar in tension. The length begins to increase and the area of the cross section at any point to decrease. An instrument called an *extensometer* is attached to the specimen so that the length AB may be read for any value of the load. The following readings were obtained from an experiment:

Diameter of rod = 0.575
Area = 0.26

In Table IV, columns I and III were read from the machine. Column II is obtained from Eq. (1) (Chap. I). For example,

$$\frac{500}{0.26} = 1{,}922 \text{ lb.}, \qquad \frac{1{,}000}{0.26} = 3{,}844, \text{ etc.}$$

The area of the cross section decreases during the stretching of the rod, but for practical purposes the area is assumed equal to 0.26 throughout the experiment.

44. Elongation.—The amount in inches that a bar is stretched by a load is the *total elongation.* Column III shows the elongation of the bar. That is, since the bar was

8 in. long originally, when the load reached 500 lb. the length was 8.0008. Then, 8.0008 − 8 = 0.0008 in. is the total elongation.

TABLE IV

Load in lbs.	Stress per sq. in.	Total Elongation	Unit Elongation from readings	Corrected Unit Elongations	Modulus of Elasticity
500	1,922	.0008	.0001	.000065	29,430,000
1,000	3,844	.00135	.00017	.000135	28,490,000
1,500	5,770	.00185	.00023	.000195	29,600,000
2,000	7,680	.00220	.00028	.000245	31,000,000
2,500	9,620	.00280	.00035	.000315	30,500,000
3,000	11,540	.00335	.00042	.000385	30,000,000
3,500	13,460	.00375	.00047	.000435	31,000,000
4,000	15,376	.00425	.00053	.000495	31,000,000
4,500	17,310	.00490	.00061	.000575	30,000,000
5,000	19,240	.00530	.00066	.000625	30,700,000
5,500	21,190	.00585	.00074	.000705	30,300,000
6,000	23,080	.00645	.00081	.000785	29,400,000
6,500	25,000	.00710	.00089	.000855	29,300,000
7,000	26,920	.00745	.00093	.000895	30,000,000
7,500	28,840	.00815	.00102	.000985	29,300,000
8,000	30,742	.00860	.00108	.001055	29,400,000
8,500	32,700	.00930	.00116	.001125	29,000,000
9,000	34,640	.00975	.00122	.001185	29,200,000
9,500	36,550	.01045	.00131	.001285	28,700,000
9,800	37,700	.04160	.00520		
9,850	37,880	.05120	.00640		
10,000	38,440	.09590	.01200		
12,000	46,200	.28590	.03574		
14,000	53,850	.45590	.05699		
16,000	61,520	.91590	.11449		
16,500	63,600	1.6059	.20074		

Unit Elongation.—If the total elongation of a bar is divided by its original length in inches, the quotient is called the *unit elongation.* It is represented by δ. For example, the values in column IV are found by dividing those in column III by 8. Thus, 0.0008 ÷ 8 = 0.0001, 0.00135 ÷ 8 = 0.00017, 0.00185 ÷ 8 = 0.00023, etc.

Using the unit stresses found in column II as ordinates (5,000 lb. to each square) and the corresponding unit elongations found in column IV as abscissas (0.00005 to each square), we plot the points *H*, *K*, *L*, *N*, *T* (Fig. 72). They are seen to lie approximately on a straight line. Draw *HM* to pass through the greatest number of points possible. It cuts the abscissa line at 0, 3.5 units to the right of *R*, the origin. This would result in a unit elongation of 0.000035 with no load. But such a condition is impossible. There must be zero elongation for zero load. That is, the true origin is at 0. There is then an error of 0.000035 lb. in each unit elongation.

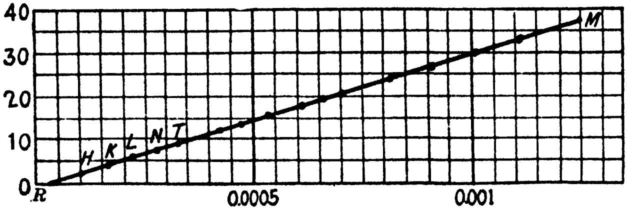


Fig. 72.

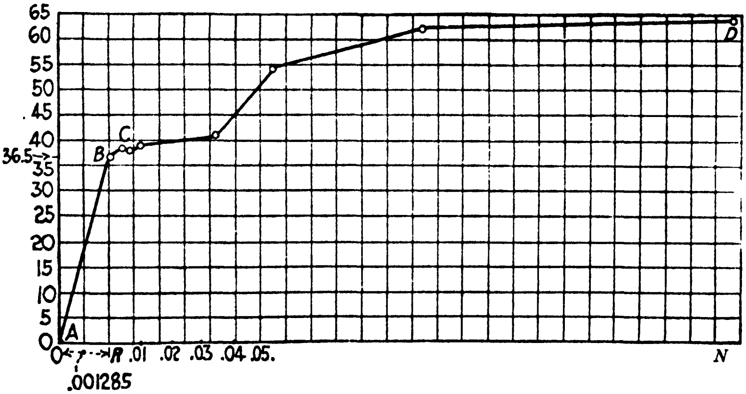


Fig. 73.—Typical stress-deformation curve for steel.

Column V contains the corrected values for the unit elongation.

The curve or line *HM* was not continued past *M* because of lack of space and also because the points would no longer follow a straight line.

In order that the relation of unit stress to unit elongation may be shown for all the points, a new curve is drawn (Fig. 73). Let *OR* = 0.001285. Then *B* is the point with stress 36,500 and unit elongation 0.001285. From *O* to *B* the points are on the straight line *OB*. From *R* to *N* a new unit for plotting elongations is taken, as shown on the diagram. The remaining stresses and elongations are plotted, the curve *ABCD* being thus obtained.

The curve from *A* to *B*, being a straight line, shows that for equal increases in unit stress there are equal increases in the unit elongation. When the unit stress reaches 36,000 lb., the curve changes direction, moving more rapidly to the right, for equal increases in unit stress. *B* is called the *elastic limit.*

If any load less than 36,000 lb. per sq. in. is applied to the bar and then removed, the bar will return to its original length. But when the stress is carried, say to 38,440 lb., and the bar elongated to an amount equal to the abscissa of point *C* and the load removed, the bar does not return to its original length. A permanent set is developed.

If more readings of the stress near *B* had been made and the values plotted carefully, a point would have been found on the curve where the stress was actually growing less while the elongation continued. This point is called the *yield point.* After passing the yield point, the stress must be increased to cause further elongation. When the stress reached 63,000, the bar was ruptured. The load at point *D* is called the *ultimate strength.*

The shape of the curve from *C* to *D* is changed in comparison with portion *ABC*, because of the change in units as mentioned before.

It is a well-known fact as established by years of experiments that the elastic limit, yield point, and ultimate strength of steel may be made to change by special methods of treatment in the manufacturing process and also by the manner in which the loads are applied. This subject is discussed in works on strength of materials and materials of engineering.

45. Modulus of Elasticity.—With reference to columns II and V of the table, as the unit stress is increased by equal amounts for equal increases in the load, the unit elongation is increased by nearly equal amounts until the unit stress reaches practically 36,000 lb. This relation is expressed in column VI, where the values are obtained by dividing each unit stress by the corresponding unit elongation. For example, 5,770 ÷ 0.000195 = 29,600,000; 15,376 ÷ 0.000495 = 31,000,000; 15,376 ÷ 0.00053 = 28,620,000; etc. After the first five, the values in column V are practically constant. The quotient obtained by dividing unit stress s by unit elongation δ is called *modulus of elasticity.* The relation is expressed as follows:

$$E = \frac{s}{\delta}\ \text{lb./sq. in.} \tag{15}$$

Since s is in pounds per square inch and δ is an abstract number, E is in pounds per square inch.

This equation shows that stress varies as unit elongation and is known as *Hooke's law.* E is the modulus of elasticity and is known as *Young's modulus.* For structural steel, E is taken equal to 29,000,000 lb. per sq. in.

With reference again to the table, it is seen that for a very slight increase in unit stress above 36,000 lb. the elongation continues very rapidly.

46. Elastic Limit.—The largest value of s for which Eq. (15) holds true is called the *elastic limit.* For soft steel, it is assumed to be about 36,000 lb.

So long as the load does not exceed the elastic limit, the bar will return to its original shape when the load is removed. If loaded beyond the elastic limit, the bar will not return to its original length and shape. It is said to have a *permanent set.*

The student must not conclude that all specimens show the same elastic limit, yield point, etc. The results depend on the character of steel tested, the testing machine, the experimenter, etc. But, after years of experimenting and manufacturing, engineers have agreed on certain values for rupturing stresses, which represent the average.

Table V contains values of the elastic limit, and E, for some building materials.

TABLE V

Material	Elastic limit in pounds per square inch	E in pounds per square inch
Structural steel	36,000	29,000,000
Hard steel	50,000	30,000,000
Wrought iron	25,000	25,000,000
Cast iron	......	15,000,000
[1]Timber		
Concrete	......	2,000,000

[1] See Appendix, Table XVII.

Thus, the moduli of elasticity of materials measure the relative ability of materials to resist deformation within the elastic limits.

47. Selection of Working Stresses.—From the foregoing articles the conclusion should be drawn that the allowable stress should be well under the elastic limit of the material. The tendency at present is to assume the allowable stress at about one-half of the elastic limit. This procedure seems to be a logical one, for the stress at which a permanent deformation takes place is now kept in mind.

48. Hooke's Law.—This law, *viz.*, unit stress varies as unit elongation, was given in Eq. (15).

$$E = \frac{s}{\delta}$$

But $s = P/A$. If λ = total elongation and l the length of the bar, then δ the unit elongation is λ/l. Substituting in Eq. (15),

$$E = \frac{s}{\delta} = \frac{P/A}{\lambda/l} = \frac{Pl}{A\lambda} \tag{16}$$

When any four of the quantities in Eq. (16) are given, the fifth may be found.

Example 1.—A steel bar 6 ft. long and 2 sq. in. in cross section is suspended from one end. Find the weight on the other end that will lengthen it 0.02 in.

Given E for steel, Table V, equal to 29,000,000.

Solution.—P = (?), $A = 2$, $\lambda = 0.02$, $l = 6 \times 12 = 72$ in.

$$\therefore 29{,}000{,}000 = \frac{P \times 6 \times 12}{2 \times 0.02}$$

$$P = \frac{29{,}000{,}000 \times 0.04}{12 \times 6} = 16{,}110 \text{ lb.}$$

Example 2.—A wrought-iron rod 2 by 4 in. and 10 ft. long is suspended vertically with a load of 20,000 lb. hanging from the bottom. Find the total elongation and the unit elongation.

Solution.—From Table V, [illegible] 25,000,000. Also, $A = 8$ sq. in., $l = 10 \times 12 = 120$ in., and $P = 20{,}000$.

$$\therefore 25{,}000{,}000 = \frac{20{,}000 \times 120}{8 \times \lambda}$$

$$\lambda = \frac{\overset{3}{\cancel{2} \times \cancel{12}}}{250 \times \underset{4}{\cancel{8}}} = \frac{3}{250} = 0.012 \text{ in.}$$

$$\delta = \frac{\lambda}{l} = \frac{0.012}{120} = 0.0001 \ \textit{Ans.}$$

Example 3.—A wrought-iron rod 50 ft. long, with a cross section of 2 sq. in., is stretched $\frac{1}{2}$ in. by a load of 44,000 lb. Find E.

Solution.—By Eq. (16),

$$E = \frac{Pl}{A\lambda}$$

But $P = 44{,}000$, $l = 50 \times 12 = 600$ in., $A = 2$, and $\lambda = \frac{1}{2}$.

By substitution,

$$E = \frac{44{,}000 \times 600}{2 \times \frac{1}{2}} = 26{,}400{,}000 \ \textit{Ans.}$$

Example 4.—Find the force necessary to punch a 1-in. hole in a ¾-in. structural-steel plate. If the punch is 4 in. long, how much is it shortened owing to the punching operation?

Solution.

a. In this problem, the material sheared from the plate is a cylinder 1 in. in diameter and ¾ in. high. The area sheared off is the convex surface of the cylinder, which is $\pi dh = \pi \times 1 \times \frac{3}{4} = 2.36$ sq. in. From Table II, the ultimate shearing stress, or stress at which steel gives way in shear, is 50,000 lb. per sq. in.

$$\therefore P = 2.36 \times 50{,}000 = 118{,}000 \text{ lb. } \textit{Ans.}$$

b. For the punch,

$$s_c = \frac{118{,}000}{\pi 1^2/4} = \frac{118{,}000 \times 4}{\pi} = 15{,}024 \text{ lb./sq. in.}$$

$$= \frac{15{,}024}{30{,}000{,}000} \times 4 = 0.002 \pm \text{ in.}$$

Example 5.—What must be the bearing area of a Washington fir post to carry safely a load of 16,000 lb. in compression? If the post is 2 ft. long, how much is it shortened by the load?

Solution.—The safe bearing or compressive stress in Washington fir is 1,600 lb. when the load is applied parallel to the grain.

Let A = the area.

Then

$$1{,}600\,A = 16{,}000 \text{ lb.}$$

$$A = 10 \text{ sq. in.}$$

Example 6.—Find the unit elongation at the elastic limit of the following materials: (*a*) structural steel, (*b*) wrought iron.

Solution.—Assume the rod to be 1 in. in length, and 1 sq. in. in cross section. Since the elastic limit for steel is 36,000 lb. per sq. in. and $E = 29{,}000{,}000$,

$$\therefore 29{,}000{,}000 = \frac{36{,}000 \times l}{1 \times \lambda}$$

$$\therefore \frac{\lambda}{l} = \delta \text{ (unit elongation)} = \frac{36{,}000}{29{,}000{,}000} = 0.00124 \text{ for steel}$$

The elastic limit for wrought iron is 25,000 whereas $E = 25{,}000{,}000$.

$$\therefore 25{,}000{,}000 = \frac{25{,}000\,l}{1 \times \lambda}$$

$$\therefore \frac{\lambda}{l} = \frac{25}{25{,}000} = 0.001 \text{ for wrought iron}$$

NOTE.—The results of this example show that steel must not be stressed so that it stretches more than 0.00124 times its length and that wrought iron must not be stretched beyond 0.001 of its length. Stretching beyond these limits would produce a permanent set.

49. Poisson's Ratio.—One of the interesting properties of materials is the deformation that takes place in a transverse direction when the piece is subjected to a force in a longitudinal direction. If a tensile force is applied, the piece will elongate and the transverse dimensions will decrease. Conversely, a compressive force will shorten the piece and increase the transverse dimensions. Owing to these transverse deformations, stresses of a corresponding nature are produced. As an average value, it may be assumed that the ratio of the transverse deformation to the longitudinal deformation is close to 0.3. This ratio is known as *Poisson's ratio.* It also follows that the stresses will be in the same ratio, for stresses are proportional to deformations.

Example 1.—A steel bar 3 sq. in. in cross section is subjected to a pull of 60,000 lb. What are the unit longitudinal and transverse deformations and unit stresses?

Solution.—$P = 60{,}000$; $A = 3$ sq. in.; $E = 29{,}000{,}000$.

$$s = \frac{P}{A} = \frac{60{,}000}{3} = 20{,}000 \text{ lb./sq. in. } \textit{Ans.}$$

$$\delta = \frac{20{,}000}{29{,}000{,}000} = 0.000689 \textit{ Ans.}$$

From these solutions, it is seen that the longitudinal tensile stress is 20,000 lb. per sq. in. The transverse stress will be 0.3 × 20,000, or 6,000 lb. per sq. in. compression. Likewise, the transverse deformation, which will be a shortening in this case, will be 0.3 × 0.000689, or 0.0002067.

Example 2.—A steel prism 2 sq. in. in cross section and 6 in. long is subjected to a load of 20,000 lb. What will be the value of the transverse stress? Poisson's ratio is 0.3.

Solution.—$P = 20{,}000$ lb.; $A = 2$ sq. in.; $E = 29{,}000{,}000$.

$$s = \frac{20{,}000}{2} = 10{,}000 \text{ lb./sq. in.}$$

$$\delta = \frac{10{,}000}{29{,}000{,}000} = 0.000345$$

$$\text{Transverse deformation} = 0.3 \times 0.000345 = 0.0001035$$

Since the stresses are proportional to the deformations, the transverse stress is equal to

$$29{,}000{,}000 \times 0.0001035 = 3{,}000 \text{ lb./sq. in. tension } \textit{Ans.}$$

The transverse stress may be obtained directly from the proportion

$$\frac{s_2}{s_1} = \frac{0.3\delta_2 \times 29{,}000{,}000}{\delta_2 \times 29{,}000{,}000} = 0.3$$

$$s_2 = 0.3s_1, \qquad \text{or} \qquad s_2 = 0.3 \times 10{,}000 = 3{,}000 \text{ lb./sq. in.}$$

50. Biaxial Stresses.—There are numerous cases where a plate is subjected to stresses at right angles to each other. The plate in a tank or a boiler affords a good illustration of such a situation, in which there is a longitudinal tension due to the pressure against the ends and a transverse tension due to the pressure in a direction perpendicular to the axis of the cylinder. It is desired to find the final stresses and possibly the deformations in the directions of the original stresses.

A portion of the boiler plate is shown in Fig. 74. The stresses shown have been computed in the usual manner. In accordance with the preceeding analysis of Art. 49, the tensile stress in a vertical direction will produce a shortening in a horizontal direction, and consequently a compressive stress which will partly offset the tension of 6,000 lb. The net stress in the horizontal direction will be

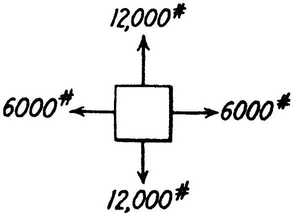


FIG. 74.

$$6{,}000 - 0.3 \times 12{,}000 = 2{,}400 \text{ lb./sq. in.}$$

Similarly, the stress of 6,000 lb. will produce a shortening of the vertical dimensions, and consequently a compressive stress which will partly offset the stress of 12,000 lb. Then the unit stress in a vertical direction will be

$$12{,}000 - 0.3 \times 6{,}000 = 10{,}200 \text{ lb.}$$

It is thus seen that the final values of the stresses are quite different from the initial stresses. The final deformations can be obtained from the final stresses.

There are various combinations of stresses. Both of the initial stresses may be compression, or they may be of opposite values. Thus it can be seen that for initial stresses of an opposite kind the final stresses will be increased. For the pressure vessel the initial stresses are always tensile.

51. Temperature Stresses.—When the temperature of a body is changed, either a shortening or a lengthening of the body will take place unless it is constrained by immovable supports. If the body is so constrained, stresses will result which are known as "temperature stresses." Frequently, in order to avoid these stresses, trusses or girders of long span are placed with one end on a roller or a sliding plate. Reinforced-concrete arch bridges are also greatly affected by temperature changes; consequently, they require a considerable amount of steel reinforcement to resist the stresses arising from this source.

The unit deformation in the length of a bar for a degree of change in temperature is called the *coefficient of linear expansion.* Let this coefficient be represented by C. If l is the original length of a bar and T the number of degrees change in temperature, the total deformation Δl is given by the relation

$$\Delta l = ClT$$

But, from Hooke's law,

$$E = \frac{s}{\delta}$$

where $\delta = \Delta l/l$, and $s = P/A$ is the temperature stress.

$$\therefore E = \frac{s}{\Delta l/l} = \frac{ls}{\Delta l} = \frac{ls}{ClT} = \frac{s}{CT}$$

$$\therefore s = ECT \qquad (17)$$

The following table contains values of C for some of the more commonly used materials:

TABLE VI

Steel, hard	$C = 0.0000074$
Steel, soft	$C = 0.0000067$
Cast iron	$C = 0.0000063$
Wrought iron	$C = 0.0000068$
Timber	$C = 0.0000028$

Example 1.—A cast-iron pipe 12 ft. long is placed between two concrete walls. Find the stress in the pipe if the temperature is raised 25°.

Solution.—$E = 15{,}000{,}000$; $C = 0.0000063$; $T = 25$.
Substituting in Eq. (17), we have

$$s = 15{,}000{,}000 \times 0.0000063 \times 25 = 2{,}362.5 \text{ lb./sq.in.}$$

52. Members Composed of Two Materials.—When a bar, beam, or rod is made from two kinds of materials and the safe load that it will carry is required, the elastic property plays an important part in the problem. A very important application of such a combination of materials occurs in reinforced-concrete construction.

Suppose that steel and wrought iron are used in a tension member. Let s_s and s_w be the stresses, respectively, and A_s and A_w the areas. Then

$$P = s_sA_s + s_wA_w \qquad (18)$$

Also,

$$E_s = \frac{s_s}{\delta} \qquad \text{and} \qquad E_w = \frac{s_w}{\delta}$$

But δ must be the same for both.

$$\therefore \frac{s_s}{E_s} = \frac{s_w}{E_w}$$

But

$$E_s = 29{,}000{,}000, \text{ and } E_w = 25{,}000{,}000$$

If maximum $s_w = 12{,}000$ lb./sq. in.,

$$s_s = \frac{12{,}000 \times 29{,}000{,}000}{25{,}000{,}000} = 13{,}920 \text{ lb./sq. in.}$$

The student must see that neither stress exceeds the allowable safe working value.

If the cross section of the bar has 4 sq. in. of steel and 3 of wrought iron,

$$P_t = 4 \times 13{,}920 + 3 \times 12{,}000 = 91{,}680 \text{ lb.}$$

53. Actual Stress.—From the discussion in Art. 49, it is seen that the transverse dimensions are decreased when a member is subjected to tension. The actual cross-sectional area will be less than the original area. If the unit tensile stress is computed by dividing the pull by the original area of the cross section, the stress will be an approximate value, which will be less than the true value.

Let d be the original diameter and $\pi d^2/4$ will be the area. For a unit stress of 29,000 lb. per sq. in., δ is 0.001 and the reduced diameter will be equal to $(d - 0.3 \times 0.001) = (d - 0.0003)$. The new area will be

$$\frac{\pi(d - 0.0003)^2}{4} = \frac{\pi}{4}(d^2 - 0.0006d + 0.0003^2)$$

The last term is negligible. If P is the total pull, the nominal stress will be

$$s = \frac{P}{\pi d^2/4}$$

and the actual stress will be

$$(\pi/4)(d^2 - 0.0006d)$$

From these expressions, it is seen that the actual stress is slightly larger than the nominal stress, but the difference is so small for stresses within the elastic limit that it is disregarded.

Above the elastic limit the rate of the reduction in area is very much increased but not in accordance with Poisson's ratio. When tests are made, the stresses are always computed by using the reduced area which is obtained from measurements of the test specimen. The reduction in area may amount to about 50 per cent of the original value.

Problems

72. Find the elongation of a 1-in. steel rod 12 ft. long under a pull of 12,000 lb. What is the unit stress?

73. A 1¼-in. steel rod 20 ft. long is subjected to a pull of 16,000 lb. What are the unit elongation and the unit stress?

74. The total elongation of a wrought-iron bar 2 in. square and 10 ft. long is 0.06 in. What is the load?

75. What is the unit elongation of hard steel when stressed to 28,000 lb. per sq. in.?

76. In an experiment in the testing laboratory, a specimen 0.82 in. in diameter and 10 in. long reached the elastic limit under a tensile load of 26,400 lb. and failed under a load of 38,320 lb. The length of the bar at its elastic limit was 10.05 in. and at failure was 13.35 in. Compute the unit stress at the elastic limit, the ultimate tensile stress, and the unit elongation for the elastic limit and for failure.

77. A platform weighing 20 tons is supported by 1-in. steel rods. How many rods are needed if they are not to elongate more than two-fifths of the elongation at the elastic limit and each supports the same amount of load?

78. A wrought-iron rod 0.6 sq. in. in cross section is elongated 0.06 in. by a load of 16,000 lb. What was the original length?

79. How much will a punch 1 in. in diameter and 3 in. long be compressed while punching a hole through a ½-in. steel plate.

80. A bar of structural steel 1¼ in. in diameter is under successive tension loads of 26,000, 32,000, and 38,000 lb. What are the unit elongations for each load? Determine which loads give stresses greater than the elastic limit.

81. A steel punch 1 in. in diameter and 4 in. long is used to punch holes through plates of various thicknesses. Can it punch a hole through a wrought-iron plate 1 in. in thickness? What maximum thickness of wrought iron can it punch? For this last case, what would be the unit deformation in the punch?

82. A wooden brace 6 sq. in. in cross section is shortened 0.06 in. by a force of 3,800 lb. What was the original length?

83. A round cast-iron post 6 ft. long, 8 in. in outside diameter, and 6 in. in inside diameter supports a load of 25 tons. What would be the total contraction?

84. A steel tie rod 24 ft. long and 1 in. in diameter is used to tie together two walls of a building. The rod is screwed up to a tension of 9,000 lb. What will be the unit stress in the rod if the temperature rises 30°F.? If it falls 30°F.?

85. A wrought-iron bar 4 in. wide and 1 in. thick is placed between two steel bars each 4 in. wide and ½ in. thick. If the bars act as a unit, what stresses are produced by a pull of 100,000 lb. applied to the combination? If the member is 5 ft. long, what is the elongation due to the load?

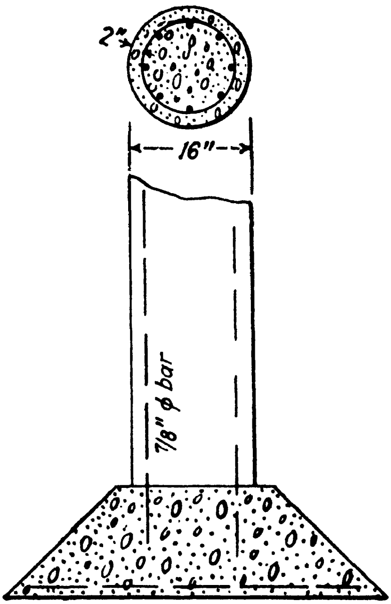


Fig. 75.

86. A concrete column (Fig. 75) supports a load of 120,000 lb. The outside diameter of the column is 16 in., and the effective diameter is 12 in. There are 10 $\frac{7}{8}$-in. round bars used for the longitudinal reinforcement. If the modulus of elasticity of concrete is 2,000,000, what is the stress in each material? If the column is 12 ft. long and if the footing contains 3 cu. yd. of concrete, what must be the side of a square footing if the soil pressure is limited to 6,000 lb. per sq. ft.

87. What space must be left between ends of steel rails, each 33 ft. long and at a temperature of 50°F., so that the ends may be in contact at a temperature of 100°F.? If these rails had been in contact at 50°F., what stress would result from the rise in temperature?

88. The effective diameter of a reinforced-concrete column is 15 in. There are 10 1-in. round rods used for the longitudinal reinforcement. What will be the stress in each material due to a load of 360,000 lb.? (The ratio of the moduli of elasticity is 15.)

89. Three 1-in. steel rods are placed parallel to the axis of a cylindrical concrete block 12 in. in diameter. The axis is vertical. What will be the unit stress and the unit deformation due to a load of 32,000 lb.? What is the transverse stress in the rods?

90. The effective diameter of a concrete column is 14 in. The longitudinal reinforcement consists of eight 1-in. round bars. What load is this column supporting if the unit stress in the concrete is 600 lb. per sq. in.? (*E* for concrete is 2,000,000.)

91. How wide a gap should be allowed between the paving slabs of concrete in a street so that the slabs will touch each other at a temperature of 90°F.? The pavement is laid at a temperature of 65°F. What would be the unit compressive stress in the slabs if the temperature rose to 110F°.?

92. A steel rod 32 ft. long is used as a hanger for a theater balcony. This rod has to support 52,000 lb. What should be the diameter of the rod if its total elongation shall not exceed 0.15 in. and the unit stress shall not exceed 12,000 lb. per sq. in.?

93. A wire made of a copper alloy is 15 ft. long and 0.04 in. in diameter. It is stretched 0.578 in. by a load of 100 lb. What is the modulus of elasticity of this alloy?

94. A flanged cylinder (Fig. 76) of 12 in. inside diameter and 6 ft. long contains steam under a pressure of 120 lb. per sq. in. How much will the 1¼-in. wrought-iron bolt be stretched by screwing up the nuts so that the heads will be steam tight?

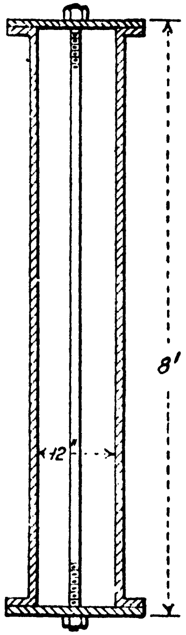


FIG. 76.

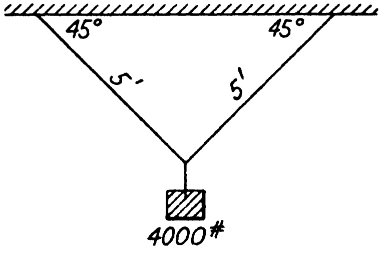


FIG. 77.

95. A load of 4,000 lb. is supported by two ½-in. steel rods, each 5 ft. long and making an angle of 45° with the vertical (Fig. 77). If the stress in each rod is 2,800 lb., what will be the elongation of each rod?

96. In Prob. 51 if member *A* is 6 ft. long, how much is it shortened owing to its stress?

97. A steel tape for measuring distances is 0.4 in. wide and 0.01 in. thick. It is 100 ft. long when subjected to a pull of 10 lb. What is the length when a pull of 50 lb. is applied?

CHAPTER IV

RESULTANT AND EQUILIBRANT OF FORCES

COMPONENTS OF A FORCE

54. Definitions.—A *body* is a collection of particles.

Concurrent Forces.—When several forces act on a body and their lines of action pass through a common point, they are called *concurrent forces.* In Fig. 78, forces F_1, F_2, and F_3 pass through the common point O and are concurrent.

Rigid Body.—When a body is acted upon by forces and the size and shape of the body is not changed, the body is said to be *rigid.* Now, all bodies are more or less elastic, and the application of a force would produce some change. But in the study of mechanics. when only the effect of forces in their tendency to produce motion is considered, the bodies on which the forces act are considered rigid.

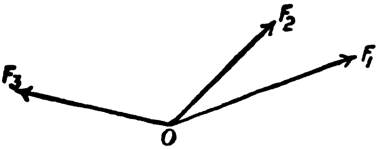


FIG. 78.

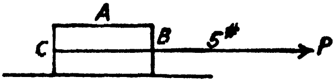


FIG. 79.

Transmissibility of a Force.—In Fig. 79, let a force P of 5 lb. be applied to a body A at the point B. The effect of P is to try to move body A and, therefore, to produce tension in the fibers of A. If the same force were applied at C in the same direction, it would cause compression in the fibers, as they resist the force. But if the body is considered rigid, then, whether the force P is applied at B or C, the same effect will be produced on A as far as motion is concerned. Therefore, a force may be considered as acting at any point on its line of action, so long as the direction and magnitude remain unchanged. This property of a force is called *transmissibility*.

Particle.—A body with dimensions practically negligible is called a *particle*. In any problem of mechanics, when the applied forces have no tendency to rotate the body on which they act, the body may be considered as a particle.

Forces acting on a particle are concurrent, the point through which they pass being the point representing the particle.

55. Types of Force System.—In the analysis of engineering problems, various types of structure must be considered. In each type the force system includes the loads together with the actions and reactions between the various members of the structure. The various force systems have certain characteristics which lend themselves to convenient classification. Thus, the forces may consist of a series of loads arranged in a line and resting on a beam. This arrangement is called a *coplanar system of parallel forces*. Likewise, a table supporting a load will represent parallel forces in space. If the lines of action of the forces intersect, they are said to be *concurrent* and may be either coplanar or non-coplanar.

56. Resultant.—When two or more forces act on a body, the single force that, acting alone, would produce the same effect on the body as the combined forces is called their *resultant*.

It is a fundamental principle of mechanics, demonstrated by experiment, that when a force acts on a body free to move the motion of the body is in the direction of the force and the distance traveled in a unit of time is proportional to the magnitude of the force.

Suppose, for example, that a force P acts on a particle A for 1 sec., causing it to move from A to B (Fig. 80). Now, suppose a second force I acts on the particle when in the position B and causes it to move to C in 1 sec. The force P ceases to act when the body has reached the position B.

Let both forces P and I act at the same time on the body in the position A. In $\frac{1}{3}$ second the body will have moved one-third the distance from A to B and also one-third BC parallel to the direction of BC. The body will then be at D. In $\frac{2}{3}$ second the particle will be at E and in 1 second at C. But triangles AMD, ANE, and ABC are similar, and $ADEC$ is a straight line. That is, when both forces act together, the particle moves from A along a straight line, *viz.*, the third side of the triangle.

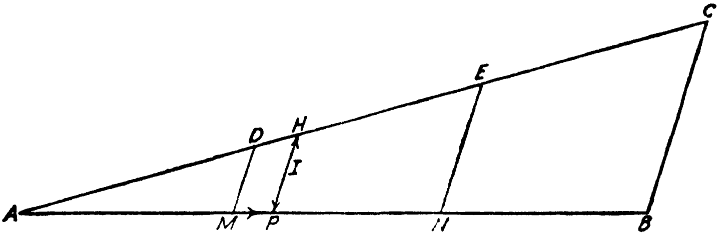


FIG. 80.

The triangle APH is similar to triangle ABC.

$$\therefore \frac{AB}{P} = \frac{BC}{I} = \frac{AC}{AH}$$

Since the distances moved by the particle in a unit of time are proportional to the forces acting, AH is a force that produces the same effect on the body as do P and I, *i.e.*, that causes it to move from A to C. Then AH is the *resultant* of P and I.

The method for determining the resultant of two concurrent forces may be stated as follows:

From any convenient point, draw a vector equal and parallel to one of the forces. From the terminal point of the first vector, draw a line equal and parallel to the second force. The closing side of the triangle drawn from the starting point of the first vector to the terminal point of the second vector is the resultant of the two forces.

Example 1.—Two forces of 30 and 40 lb., respectively, act on a body, making an angle of 90° with each other (Fig. 81). Find the resultant.

Solution.—From any point A, draw AB equal to 30 lb., to any convenient scale and parallel to force P.

From B draw a line BC, perpendicular to AB and in length to represent 40 lb. Then the line AC is the resultant in magnitude and direction. By measurement, AC is found to be 50 lb.

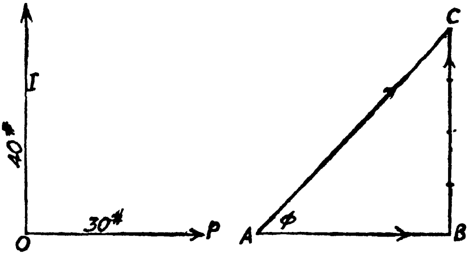


FIG. 81.

Notice that since the resultant forms a triangle with the two given forces an algebraic solution of the triangle will give AC. Since ABC is a right triangle,

$$AC = \sqrt{\overline{AB}^2 + \overline{BC}^2} = \sqrt{900 + 1{,}600}$$
$$\therefore AC = 50 \text{ lb.}$$

The angle ϕ that AC makes with AB may be found from the tangent.

$$\tan \phi = \frac{BC}{AB} = \frac{40}{30} = 1.3333$$
$$\therefore \phi = 53°8'$$

Example 2.—Find the resultant of two forces of 50 and 75 lb., respectively, acting on a body at an angle of 50° with each other.

Example 3.—Two forces of 100 lb. each act on a body at an angle of 120° with each other. Find the resultant in magnitude and direction by constructing the triangle, and also by trigonometry.

Example 4.—Two equal forces P act on a body at an angle of 60° with each other. Find the resultant by trigonometry.

57. Equilibrium.—*When the resultant of any number of forces acting on a body is zero, the forces are said to be in "equilibrium."*

The student should notice that when forces are in equilibrium the body on which they act may be at rest or in uniform motion.

Statics is a study of forces in equilibrium.

When two forces are acting on a body, the third force that will hold them in equilibrium is called the *equilibrant* or the *balancing force*.

The relation between three forces in equilibrium is best illustrated by an experiment. In Fig. 82, let *OA*, *OB*, and *OC* be fastened to a ring at *O*. Spring balances are fastened at *A*, *B*, and *C*. The springs are then stretched and the rings fastened to pegs *E*, *F*, and *G*.

Balances *E*, *F*, and *G* show forces of 12, 15, and 20 lb., respectively. These forces are in equilibrium because the ring *O* is at rest.

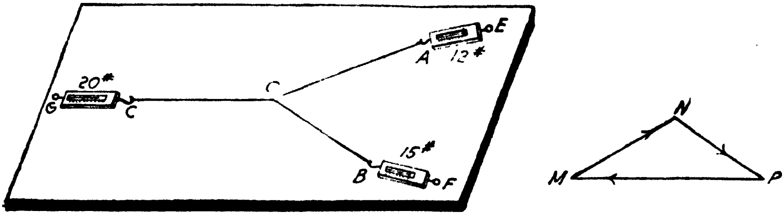


Fig. 82.—Experiment to illustrate the triangle law.

Now draw *MN* parallel to *OA* and to scale equal to 12 lb. From *N*, draw *NP* equal to 15 lb. and parallel to *OB*. Join *P* and *M*. *PM* by careful measurement will be found to scale 20 lb. If the angles of the triangles *MNP* are found by a protractor and compared with angles *AOC* and *BOC*, *MP* will be found parallel to *OC*. The equilibrant, or balancing force, of any two forces is the closing line of the triangle constructed as in Art. 56, with the arrowhead pointing in such a direction that the arrowheads of all the forces appear to follow each other around the triangle.

That this relation should be true is easily seen from Art. 56. The resultant of any two forces, such as *OA* and *OB* of Fig. 82, is the closing line *MP*. Now two forces can be in equilibrium only when they act in the same straight line but are opposite in direction, by Art. 8; *i.e.*, the balancing force of *MP* is *PM*.

Triangle *MNP* is called the *force triangle*.

Theorem.—Three concurrent forces in equilibrium may be represented as sides of a triangle, called the force triangle.

58. Principle of Concurrence.—If three forces acting on a body are in equilibrium, their lines of action must meet in a common point. As already shown, the resultant of any two concurrent forces is a single force acting through their point of intersection. When equilibrium exists, this resultant force is balanced by a third force. But, in order to be in equilibrium, these two forces must act in the same straight line. To meet this condition, the lines of action of the original forces must meet at a point. This principle of concurrence will be found useful in later problems in which the direction of some of the unknowns is not at first evident.

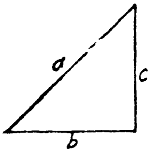


FIG. 83.

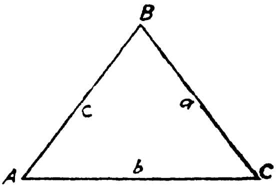


FIG. 83a.

59. Methods of Solving.—Since three concurrent forces in equilibrium may be represented as a triangle of forces, the solution of such a problem, in mechanics, is the solution of a triangle

If the triangle of forces is constructed accurately, the unknown parts of the triangle may be found by measurement. This is called the *graphical solution.*

When the triangle is solved by means of algebra or trigonometry, the method is called *algebraic.*

The three theorems used in the algebraic method are as follows:

1. *In any right triangle the square of the hypotenuse is equal to the sum of the squares of the two legs.*

$$a^2 = c^2 + b^2 \qquad (19)$$

2. *Law of sines. In any triangle the sides are to each other as the sines of the opposite angles.*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (20)$$

3. *Law of cosines. In any triangle, the square of any side is equal to the sum of the squares of the other two sides, minus twice the product of the sides and the cosine of their included angle.*

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad (21)$$

60. Resultant of Two Forces Algebraically.—Let P and Q be two forces making an angle θ with each other (Fig. 84). Draw MN equal and parallel to Q. Join O and N.

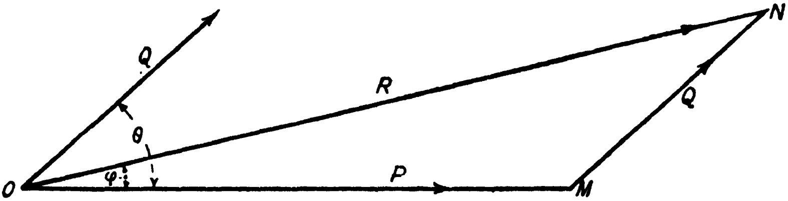


FIG. 84.

Then ON is the resultant R. In the triangle OMN the side R is unknown, but the two sides P and Q and their included angle are known. By the law of cosines,

$$R^2 = P^2 + Q^2 - 2PQ \cos \angle OMN$$

But,

$$\angle OMN = 180° - \theta$$

Also,

$$\cos \angle OMN = \cos (180 - \theta) = -\cos \theta$$
$$R^2 = P^2 + Q^2 + 2PQ \cos \theta \qquad (21a)$$

Example 1.—Given force $P = 50$ lb., $Q = 40$ lb., and $\theta = 30°$. To find R.

Example 2.—In Eq. (21*a*), let $\theta = 0°$, also $\theta = 180°$. Find R in each case, and compare with Art. 8.

Example 3.—Show that when $\theta = 90°$, Eq. (21*a*) becomes (19) of Art. 59.

The angle φ that R makes with P is found by the law of sines, Eq. (20), Art. 59.

$$\frac{R}{\sin \angle OMN} = \frac{Q}{\sin \varphi}$$

But $\sin \angle OMN = \sin \theta$.

$$\therefore \sin \varphi = \frac{Q \sin \theta}{R}$$

where θ is the *angle* between the forces.

If θ equals 90°, $\sin \theta$ equals 1.0, and $\sin \theta$ equals Q/R which is the ratio of the opposite side to the hypotenuse. cos 90° equals 0, and Eq. (21*a*) reduces to

$$R^2 = P^2 + Q^2$$

since the last term reduces to zero. This equation is now the same as (19).

Example 4.—Two forces of 75 and 100 lb., respectively, make an angle of 110° with each other. Find their resultant and the angle it makes with the 100-lb. force.

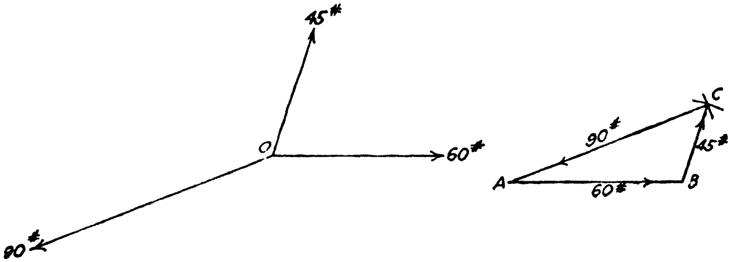


FIG. 85.

Example 5.—Three concurrent forces of 45, 60, and 90 lb., respectively, are in equilibrium (Fig. 85). Find the angles that these forces must make with each other.

Solution.—Since these forces are in equilibrium, they may be represented as the three sides of a triangle.

With A as a starting point, draw AB parallel and equal to the 60-lb. force. With B as a center and a line equal to 45 lb. as a radius, strike an arc. With A as a center and a radius equal to 90 lb., draw an arc intersecting the first at C. Draw AC and BC. Then ABC is the required force triangle with sides parallel to the given forces. The required angles could be found by means of a protractor. A more accurate method is to apply the law of cosines.

$$\overline{90}^2 = \overline{60}^2 + \overline{45}^2 - 2 \times 60 \times 45 \cos B$$
$$\cos B = -\frac{2{,}475}{5{,}400} = -0.4583$$
$$\therefore \angle B = 117^\circ 17'$$

Also,

$$\overline{45}^2 = \overline{60}^2 + \overline{90}^2 - 2 \times 60 \times 90 \cos A$$
$$\cos A = \frac{9{,}675}{10{,}800} = 0.8954$$
$$\therefore \angle A = 26^\circ 23'$$

Then

$$\angle C = 180^\circ - [117^\circ 17' + 26^\circ 23'] = 36^\circ 20'$$

The angle between the forces is easily seen to be the supplement of the angles of the triangle.

Thus the angle between 45-lb. and 60-lb. forces is

$$180^\circ - 117^\circ 17' = 62^\circ 43'$$

61. Free Body.—In Chap. I, force was defined as the action of one body upon another. In solving problems of mechanics, it is often convenient to disregard bodies acting upon a single body and to replace these actions by force vectors, having their magnitude and direction the same as those of the actions produced by each of the acting bodies. A body thus represented is said to be a "free" body.

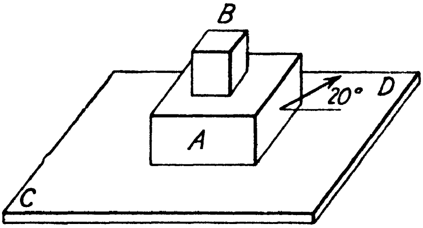


FIG. 86.

It is very necessary to designate clearly the free body and to account for all the forces that act on it. A sketch showing the body with the forces acting is an important aid in the investigation and solution of problems and should always be drawn.

Take, for example, a body weighing 200 lb., resting on a rough floor, and acted upon by a force of 20 lb. as shown in

Fig. 86. Another body weighing 50 lb. rests on the first body. The forces are in equilibrium.

Body A, considered as a free body and a particle, is acted upon by forces as shown in Fig. 87. A force of 50 lb. acting downward represents the action of the body B; 200 lb. the action of gravity on A; N upward, the normal reaction of the table, F acting to the left, represents the frictional resistance of the table to motion, and 20 lb. to the right. This figure shows all the forces acting on A.

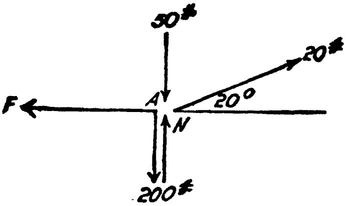


FIG. 87.

Example.—A body weighing 500 lb. is suspended from a beam by two ropes making angles of 20 and 30° with the vertical, respectively. Find the tensions in the ropes.

Solution.—Let AB and BC be the ropes that support body B (Fig. 88). AB and BC are two force pieces and therefore tension members.

Set point B out as a free body. There are three forces, *viz.*, 500 lb. down, and F_1 and F_2 up, at angles 20 and 30° as shown.

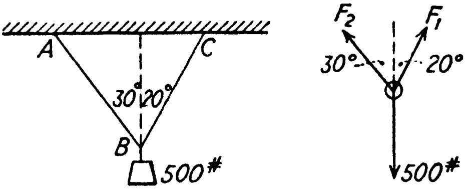


FIG. 88.

From any point O in Fig. 89, draw OD to represent 500 lb. From D, draw a line parallel to F_1 but indefinite in length. F_1 is known in direction but not in magnitude. Since the three forces must form a triangle, force F_2 must end at the point O. Therefore, draw a line through O, parallel to F_2, cutting the second line at H. Triangle ODH is the force triangle, and DH and HO to scale represent F_1 and F_2. These values may be obtained by direct measurement.

By the law of sines,

$$\frac{F_1}{500} = \frac{\sin 30°}{\sin 130°}, \qquad \frac{F_2}{500} = \frac{\sin 20°}{\sin 130°}$$

$$F_1 = \frac{500 \times 0.5}{0.7660} = 326 \text{ lb.}, \qquad F_2 = \frac{500 \times 0.342}{0.7660} = 223 \text{ lb.}$$

62. Analysis of Simple Structure.—A *Simple structure* is a body formed of tension and compression members, hinged or pinned together. The hinges or pins are assumed frictionless, and the weight of the parts is neglected. The external forces are assumed to act at the joints. Most roof trusses and many bridge trusses are treated as simple structures. At each joint, there are three or more concurrent forces, the directions of which are known. Since these forces are in equilibrium, the joint may be set out as a free body. Also, the entire simple structure is in equilibrium under the action of external forces and may be treated as a free body by methods to be developed in future articles.

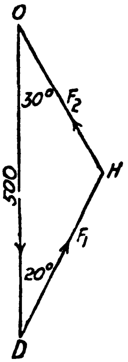


Fig. 89.

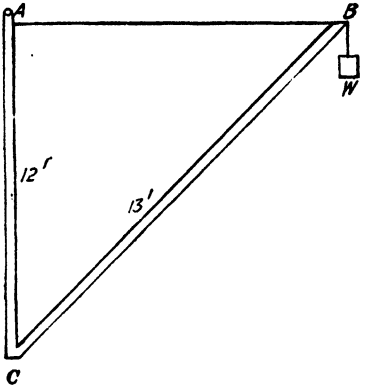


Fig. 90.

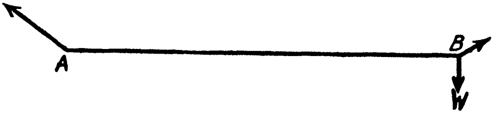


Fig. 91.

Take a structure as shown in Fig. 90. The tie rod AB is held at A and B by pins at which points external forces are exerted. Now, let AB be set out as a free body (Fig. 91).

At B, the pin is pulled downward because of the weight W. There is also a push upward and to the right due to

CB, for its function is to push up on the load. The tendency of the load W is to swing to the right and down. This motion is prevented by AB, through the pin at A. Now, the two concurrent forces at B may be replaced by their resultant. Then AB is in equilibrium under the action of two forces, one at A and one at B (call them R_A and R_B). From Art. 8, this condition is true only *when these two forces are equal and opposite in direction, i.e., when they are in the direction of the bar* (Fig. 91*a*). AB is a two-force piece, having forces acting at but two points. It is a tension member. Similarly, CB set out as a free body would be shown to be a two-force piece or compression member (Fig. 91*b*). Therefore, ABC is a simple structure, and the direction of the stresses in each member is known. A vertical post carrying a load, a hoisting cable, a jack placed under the axle of a truck, a brace or prop—these, in general, are illustrations of two-force pieces. It is typical of the two-force member that it is acted on by forces at its ends only. The resultant of the forces at either end must act against the resultant at the other end. In order to do so, the forces must act along the axis of the piece. The lines of action must be along the same straight line, and the forces will produce either tension or compression.

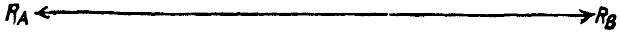


FIG. 91*a*.

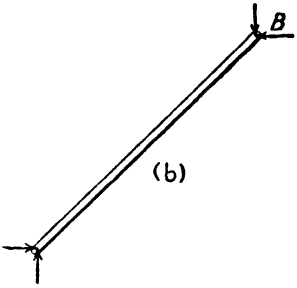


FIG. 91*b*.

Example.—Let $W = 4{,}000$ lb., $AB = 5$ ft., $AC = 12$ ft. Then $BC = 13$ ft., for ABC is a right triangle. Find the stresses in AB and BC.

Solution.—Take pin B as a free body with 4,000 lb. and F_1 and F_2 the concurrent forces acting on the pin (Fig. 92). From any point O, draw OM equal to 4,000 lb. by scale. From M draw a line parallel to F_2 and indefinite in length. Since the figure must close, F_1 terminates at O. Then from O, draw a line parallel to F_1 to intersect the second line drawn at point N. Then $MN = F_2$, and $NO = F_1$. F_1 and F_2 may be determined by direct measurement (Fig. 92a).

The force triangle gives the direction of the force as well as its magnitude. If the arrowhead is put on CB near B, pointing toward N, it shows that CB exerts a thrust, or is in compression, whereas AB is seen to be in tension.

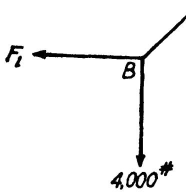


Fig. 92.

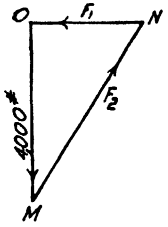


Fig. 92a.

Since MNO is a right triangle, F_1 and F_2 may be found by simple trigonometry.

$$\cos M = \frac{4{,}000}{F_2}$$

From Fig. 90,

$$\cos M = \cos C = \tfrac{12}{13}$$

$$\therefore \frac{12}{13} = \frac{4{,}000}{F_2}$$

$$F_2 = \frac{13 \times 4{,}000}{12} = 4{,}333 \text{ lb.}$$

$$\frac{F_1}{4{,}000} = \tan M = \frac{5}{12}$$

$$\therefore F_1 = \frac{4{,}000 \times 5}{12} = 1{,}666 \text{ lb.}$$

Another very useful method is suggested from similar triangles. By construction the force triangle MNO is similar to CBA of the original structure. Then the homologous sides are proportional, or

$F_2:13::4{,}000:12, \quad \therefore F_2 = 4{,}333$ lb.
$F_1:\ 5::4{,}000:12, \quad \therefore F_1 = 1{,}666$ lb.

Whenever the principle of similar triangles is applicable, it should be used.

63. Components of a Force.—In Art. 56, the resultant of two forces was defined as a single force that will produce the same effect as the two forces. Also, it was stated that the two forces, with their resultant, form a triangle.

The converse of this theorem is also true, that a force may be replaced by any two forces which with the given force form a triangle. In Fig. 93, AC and CB are two forces that, if applied to a body at A, produce the same effect as AB. AC and CB are called the *components* of AB. Note the direction of the arrows.

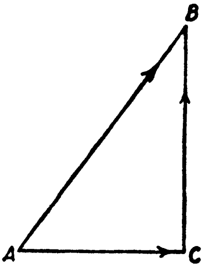


Fig. 93.

The student should notice that a force may be resolved into any number of pairs of components, for, on any line as a side, any number of triangles may be constructed. However, as soon as the directions of the sides are definitely known or assigned, then but one triangle can be constructed. There is then but one pair of components of the force.

Example 1.—Resolve a force of 100 lb. into two components making angles of 30 and 70°, respectively, with the force.

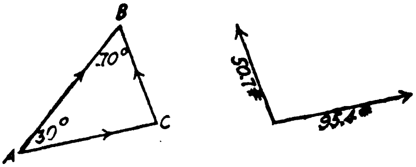


Fig. 94.

Solution.—Let AB be the given force (Fig. 94). Draw AC and BC making angles of 30 and 70°, respectively. Then AC and CB are the components. By the law of sines,

$$\frac{AC}{100} = \frac{\sin 70^\circ}{\sin 80^\circ}, \qquad 1C = 95.4 \text{ lb.}$$

$$\frac{CB}{100} = \frac{\sin 30^\circ}{\sin 80^\circ}, \qquad CB = 50.7 \text{ lb.}$$

Therefore forces of 95.4 lb. and 50.7 lb. acting at A are the components of AB.

Example 2.—Resolve a force of 60 lb. into two components, one of which is 38 lb. and makes an angle of 45° with the given force. Hint.—Construct the triangle, and apply cosine and sine laws.

Example 3.—A force of 93 lb. has forces of 60 lb. and 75 lb. for components. Find their directions with the given force.

Example 4.—A force of 100 lb. has components making angles of 30 and 105°, respectively, with the force. Find their magnitudes.

Problems

98. A weight of 6,000 lb. is suspended from the point C (Fig. 95). Find the stress in AC and BC if $\angle WCA = 30°$ and $\angle WCB = 60°$. Which member is a tie rod, and which member is a brace?

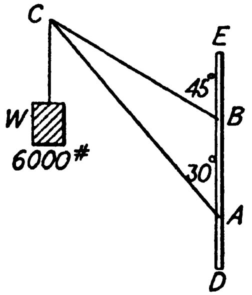


FIG. 95.

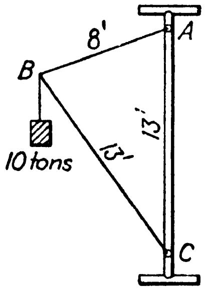


FIG. 96.

99. A load of 10 tons is carried by the simple derrick shown in Fig. 96. By making use of similar triangles, find the stresses in AB and BC.

100. A body weighing 160 lb. is suspended from a hook by a rope 12 ft. long. If a horizontal force is applied to the body so as to swing it to one side and cause the rope to make an angle of 30° with the vertical, what will be the tension in the cord, and what is the amount of the horizontal force?

64. Rectangular Components of a Force.—When a force is resolved into two components that are at right angles to each other, they are called *rectangular components*. These components with the given force as hypotenuse form a right triangle.

In the solution of many problems in mechanics, it is customary to choose an origin and x and y axes at right angles to each other. The force or forces acting are then resolved into their rectangular components along these axes. If the x axis is horizontal and the y axis vertical, the components of a force along these axes are often called ***horizontal*** **and** ***vertical components.***

The axes are not always horizontal and vertical; but the components will be referred to as x and y components and represented by F_x and F_y, when the given force is F.

Let F be a force making an angle Θ with the horizontal as shown in Fig. 97. From the point B, drop a perpendicular to OA. Then $F_x = OA$, and $F_y = AB$.

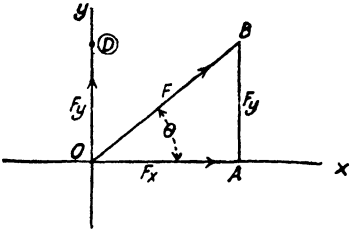


Fig. 97.

From trigonometry: $\cos \Theta = F_x/F$, and $\sin \Theta = F_y/F$; or $F_x = F \cos \Theta$, and $F_y = F \sin \Theta$.

But F_x *is the projection of* F *on the* x *axis, and,* since $AB = OD$, F_y *is the projection of* F *on the* y *axis.*

The following definitions may then be given:

The x *component of a force is the force times the cosine of the angle that the force makes with the* x *axis.*

The y *component of a force is the force times the cosine of the angle that the force makes with the* y *axis, or the force times the sine of the angle that the force makes with the* x *axis.*

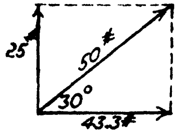


Fig. 98.

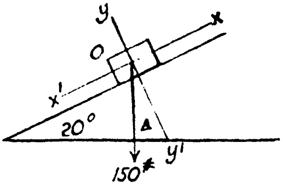


Fig. 99.

Example 1.—A force of 50 lb. makes an angle of 30° with the horizontal. Find F_x and F_y (Fig. 98).

Solution.

$$F_x = F \cos \Theta = 50 \cos 30° = 43.3 \text{ lb.}$$
$$F_y = F \sin \Theta = 50 \sin 30° = 25 \text{ lb.}$$

Example 2.—A body weighing 150 lb. rests on an inclined plane making an angle of 20° with the horizontal. Find F_x and F_y of the weight (Fig. 99).

Solution.—Take the x axis parallel and the y axis perpendicular to the plane.

The weight of the body is a force of 150 lb. acting down.

But $\angle X'OA = 70°$ and $\angle AOY' = 20$ deg.

Then

$$F_x = 150 \cos 70° = 150 \times 0.342 = 51.3 \text{ lb.}$$
$$F_y = 150 \cos 20° = 150 \times 0.9397 = 141 \text{ lb.}$$

Example 3.—Find the rectangular components of a force of 200 lb. making an angle of 10° with the x axis; making an angle of 35°; an angle of 55°; an angle of 80°; an angle of 90°.

When a force is at right angles with a line, its component is zero along the line; that is, the force has no effect on a body in a direction at right angles to the force.

65. Inclined Plane.—Let a body of weight W rest on an inclined plane making an angle Θ with the horizontal (Fig. 100).

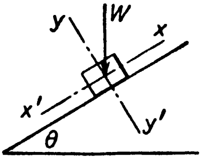


FIG. 100.

Choose the axes along and perpendicular to the face of the plane. Then, by Art. 64,

$$F_x = W \sin \Theta$$
$$F_y = W \cos \Theta$$

That is, *the component of the weight of a body parallel to the plane is equal to the weight times the sine of the angle that the plane makes with the horizontal.*

The component of the weight perpendicular to the plane is the weight times the cosine of the angle that the plane makes with the horizontal.

The student should master and memorize these principles because of their importance in problems in mechanics.

Since the y component is the one perpendicular to the plane it produces no motion of the body.

The component of the weight parallel to the plane is the force that acts to drag the body down the plane. If the body does not move, it is because of the reaction of some external force such as friction,[1] or of an applied force.

If there is no friction, then it can be seen that the applied force necessary to prevent the body from sliding down the plane will be equal to $W \sin \Theta$. A force parallel to the plane and slightly greater than $W \sin \Theta$ by the amount of the friction encountered will be needed to move the body up the plane. Frequent use is made of the inclined plane in loading heavy machinery on trucks, in jackscrews, and in ordinary screw threads. The general principle involved is that a relatively small force exerted parallel to the plane is capable of displacing the body along the plane and thereby raising the body to the desired height.

---

[1] The force that acts between two surfaces in contact and tends to resist motion of one surface over the other is called *friction*. A more complete discussion of friction is found in a later chapter.

Example 1.—An inclined plane is formed by using a 12-ft. plank. The floor of a truck is 3 ft. above the ground. It is desired to load a barrel of salt weighing 300 lb. by rolling it up the 12-ft. plane. How much force must be exerted parallel to the plane?

Solution.

$$\sin \Theta = \tfrac{3}{12} = \tfrac{1}{4}$$
$$W \sin \Theta = 300 \times \tfrac{1}{4} = 75 \text{ lb. } \textit{Ans.}$$

A force slightly in excess of 75 lb. and applied to the barrel in a direction parallel to the plane will roll the barrel up the plane. It is seen that a vertical lift without using the inclined plane would require 300 lb.

Example 2.—Find the components parallel and perpendicular to the plane of a 300-lb. body resting on the plane, with the following angles of inclination: 20, 40, 70, and 85°.

66. Resultant of More than Two Forces in a Plane.—Suppose that three or more forces act through a common point O, as in Fig. 101. To find the resultant.

a. Graphical Solution.—From any convenient point A, in Fig. 102, draw AB parallel and equal to OF_1. From B, draw BC equal and parallel to OF_2. By Art. 56, AC is the resultant of OF_1 and OF_2. From C, draw CD equal and parallel to OF_3. AD is the resultant of AC and CD. It is, therefore, the resultant of OF_1, OF_2, and OF_3. From D, draw DE equal and parallel to OF_4. AE, the resultant of AD and DE, is then the resultant of the four forces OF_1, OF_2, OF_3, and OF_4. *ABCDEA is the polygon of forces.*

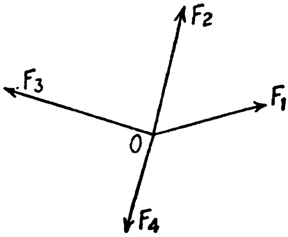


Fig. 101.

The method for determining the resultant of any number of concurrent forces is as follows: *From any point, construct the force polygon by adding the vectors taken in order, either clockwise or counterclockwise around the point. The closing line of the polygon drawn from the starting point of the first vector to the terminal point of the last vector is the resultant of the concurrent forces.* (Notice the direction of the arrowheads.)

Example.—Given concurrent forces of 20, 30, 10, and 40 lb. making angles of 10, 30, 120, and 125° with a horizontal line through the common point. Find the resultant.

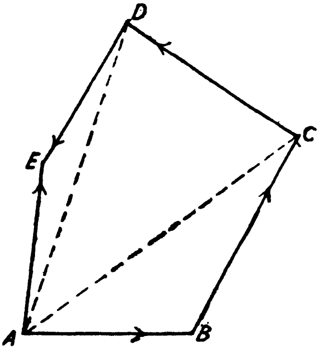


FIG. 102.

67. Equilibrium of More than Two Forces.

—When three or more forces are in equilibrium, their resultant is zero. From Fig. 102, the resultant of the four forces shown is AE. The single force to be in equilibrium with AE is EA, a force equal in magnitude, and opposite in direction. But EA closes the polygon. *Then, when concurrent forces are in equilibrium, if the force diagram is constructed it must be a closed polygon.*

Example 1.—A body weighing 50 lb. rests on a rough horizontal plane. A force of 10 lb. acts horizontally to the right. The body does not move. Find the force-resisting motion, and the reaction of the plane (Fig. 103).

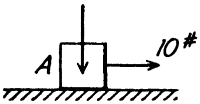


FIG. 103.

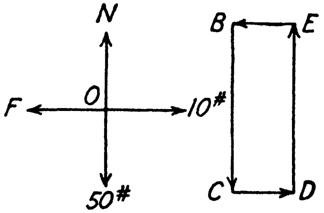
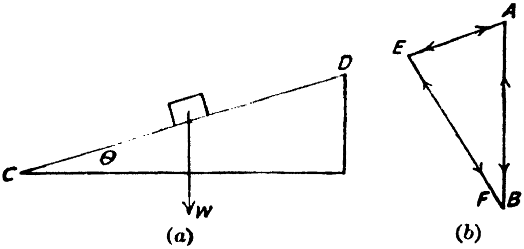


FIG. 103a.

Solution.—Let the body be considered as a particle and set out as a free body. Let O be this point. At O, the weight of 50 lb. acts down. The reaction of the plane is up and is represented by *N*. The force of 10 lb. acts to the right. Since the body does not move, there must be a force acting between the two surfaces in contact to oppose the motion. Call it *F*. Construct the force polygon *BCDEB*, taking the forces in order, counterclockwise. Since *N* and *F* are known in direction only, first draw the forces known completely. *BC* is equal to 50 lb., and *CD* to 10 lb At *D*, erect a perpendicular indefinite in length. Since the four forces are in equilibrium, they must form a closed polygon. Then force *F* must end at *B*. From *B* draw a line perpendicular to *BC* to meet the perpendicular from *D* at the point *E*. Then *DE* is *N*, and *EB* is *F*. Since *BCDE* is a rectangle, lines *BC* and *DE* are equal in magnitude, or *DE* = 50 lb. and *EB* = 10 lb.



(*a*) (*b*)

FIG. 104.

Example 2.—A body weighing *W* lb. rests on a plane inclined at an angle of θ with the horizontal. Find the reaction of the plane, assuming that the plane is rough enough to prevent motion (Fig. 104).

Solution.—The weight of the body is a force that acts vertically toward the center of the earth. Since the body is in equilibrium under the action of the two forces, *viz.*, the weight *W* and the reaction of the plane *R*, they must be equal in magnitude and opposite in direction. Let *AB* represent *W*. Then *BA* is *R*. But from Art. 63, it was seen that a force can be resolved into two components along any assumed directions. Let the directions of the components be along the face of the plane and perpendicular to the plane. Then *AE* and *EB* are the rectangular components of *AB*. Also, *BE* and *EA* are the components of *BA*. Now *EB* and *BE* are perpendicular to the plane and have no effect on the body so far as motion is concerned. *AE* is the component tending to drag the body down the plane. Its equilibrant *EA* is the force opposing motion.

The reaction of the plane due to the weight is called the *normal reaction of the plane* and the component along the plane is called *friction*. It follows that when a surface is perfectly smooth, since friction is zero, *the reaction is normal to the surface against which the body presses.*

Example 3.—A bar AC pinned to the wall at A with a load of 1,000 lb. at C is held in a horizontal position by the strut, or compression member DB, pinned at D and B. To find the stress in DB and the pin reaction at A (Fig. 105).

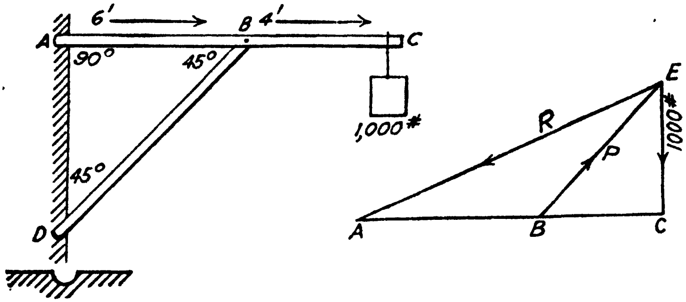


FIG. 105.

Solution.—The bar AC is in equilibrium under the action of three forces, at A, B, and C, respectively. By Art. 57 and the principle of concurrence, they must meet in a point. The 1,000 lb. is vertical. Since DB is a two-force piece, or compression member, the direction of the force at B is known. Prolong these forces to meet at E. Then the pin reaction at A is a force that must pass through E. The force triangle may now be constructed and the forces R and P found. (This problem is left for the student.) Check the graphical solution by solving the triangle, using the law of sines.

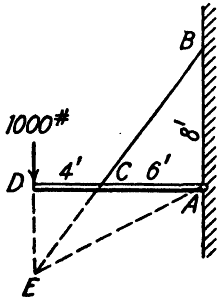


FIG. 106.

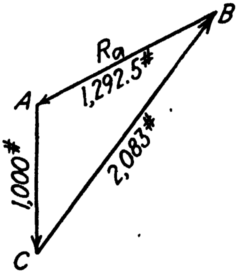


FIG. 106*a*.

Example 4.—The beam AD is pinned to the wall at A and held in a horizontal position by the tension member BC. There is a load of

1,000 lb. at *D*. What are the stress in *BC* and the reaction at *A* (Fig. 106)?

Solution.—The beam *AD* is in equilibrium under the action of three forces, at *A*, *C*, and *D*, respectively. Since the member *BC* is acted on by forces at its ends only, this member is a two-force piece. The force at *B* is in the direction of *BC*. The load is vertical. Prolong line *BC* to intersect the vertical through *D*. Then *E* will be the point of concurrence of the forces. The reaction at *A* is a force of which the line of action must pass through *E*. The directions of all the forces are now known, and the force triangle can be constructed. This is shown in Fig. 106*a* and can be solved either graphically or trigonometrically by the law of sines.

Scaling the figure, we have BC = 2,083 lb., R_a = 1,292.5 lb.

68. Resultant of Concurrent Forces by Summation.

—Any force may be resolved into two components at right angles to each other. If x and y axes are taken through the point common to all the forces and each force is resolved into its x and y components, the given forces are then replaced by two sets of forces acting along the axes of reference. The forces or component forces along the x axis may be combined by ordinary addition (regard being given to the sign) into a single force. This force is represented by ΣX. Similarly, there is found a force acting along the y axis and represented by ΣY (Fig. 107). The hypotenuse of the right-angled triangle with legs equal to ΣX and ΣY is the

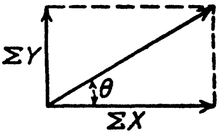
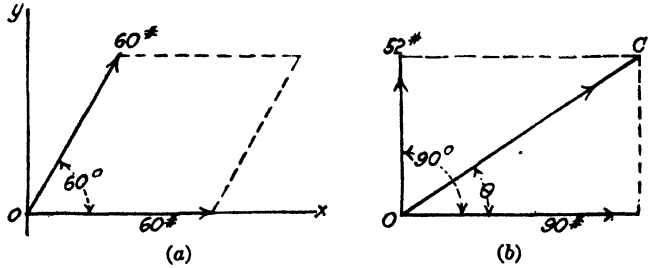


FIG. 107.



(*a*) (*b*)

FIG. 108.

resultant force in magnitude and direction of the given system of forces. That is

$$R = \sqrt{\Sigma X^2 + \Sigma Y^2} \qquad \theta = \tan^{-1}\left(\frac{\Sigma Y}{\Sigma X}\right) \tag{22}$$

Example 1.—Two forces of 60 lb. each, one of which is horizontal and the other at an angle of 60° with the horizontal, act on a body. Find their resultant (Fig. 108).

Solution.—Let Ox and Oy be the axes of reference. The components may be tabulated as follows:

Force *lb.*	∠ with x axis *degrees*	x comp. *lb.*	y comp. *lb.*
60 60	0 60	60 30	0 52
		90	52

Then

$$\Sigma X = 60 + 30 = 90 \text{ lb.}$$
$$\Sigma Y = 0 + 52 = 52 \text{ lb.}$$

In Fig. 108*b*, ΣX and ΣY are shown at 90° with each other. Their resultant is the hypotenuse OC. But

$$OC = \sqrt{\overline{90}^2 + \overline{52}^2} = 103.9 \text{ lb.} = R$$
$$\tan \theta = \tfrac{52}{90} = 0.557 \qquad \therefore \theta = 30°$$

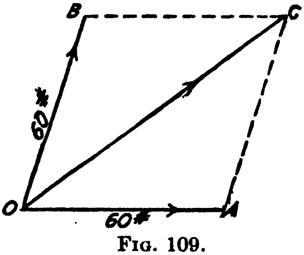


Fig. 109.

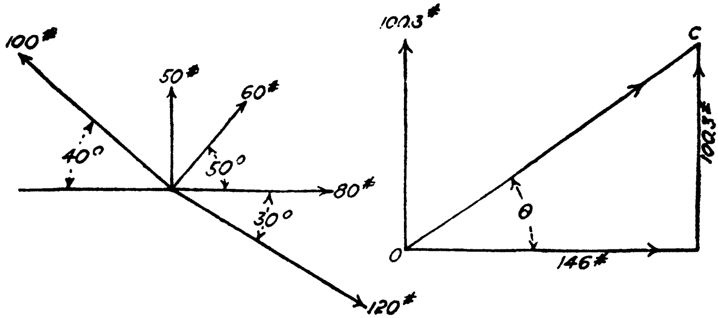
Then a force of 103.9 lb. acting at an angle of 30° with the horizontal is the resultant of the given forces.

That the angle of the resultant is 30° could have been seen by constructing the parallelogram of forces as in Fig. 109. Since $OA = OB$, the figure is equilateral and therefore the diagonal bisects the angles O and C.

If the body at O were a pin fastened to a vertical wall and held rigid, the pin would exert a force equal to OC in the opposite direction, by the law that action equals reaction.

Therefore, the pin reaction is a force of 103.9 lb. acting from O, opposite to OC, the angle being 30° below the negative end of the x axis.

Example 2.—Find the resultant of the forces with magnitude and direction as shown in Fig. 110.



(a) (b)

FIG. 110.

Solution.—Resolve each force into its x and y components and arrange in a table.

Force *lbs.*	With x axis *degrees*	x comp. *lbs.*	y comp. *lbs.*
80	0	+ 80	+ 0
60	50	+ 38.6	+ 46
50	90	+ 0	+ 50
100	40	− 76.6	+ 64.3
120	30	+104	− 60
		+146	+100.3

Notice that the forces in column 1 cannot be added by arithmetic because they do not act in the same straight line, whereas those in columns 3 and 4 are added algebraically because they do act in the same line.

$$\Sigma X = 80 + 38.6 + 0 - 76.6 + 104 = +146 \text{ lb.}$$
$$\Sigma Y = 0 + 46 + 50 + 64.3 - 60 = +100.3 \text{ lb.}$$

Then from Fig. 110*b*, the resultant *Oc* or

$$R = \sqrt{\overline{146}^2 + \overline{100.3}^2} = 177 \text{ lb.}$$

$$\theta = \tan^{-1}\left(\frac{100.3}{146}\right) = 34°29'$$

Therefore, the resultant is a force of 177 lb. acting at an angle of 34°29′ with the horizontal.

69. Equilibrium of Concurrent Forces.—In Art. 68, it was found that, when concurrent forces act on a body, the resultant

$$R = \sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Y}^2}$$

Now, when forces are in equilibrium, their resultant is zero. Then

$$\overline{\Sigma X}^2 + \overline{\Sigma Y}^2 = 0$$

But both expressions are plus, being squares of numbers. The sum of two positive numbers cannot be zero unless each one is zero. That is,

$$\Sigma X = 0 \qquad \text{and } \Sigma Y = 0 \qquad (23)$$

Then, when concurrent forces are in equilibrium, the sum of the components along each of two rectangular axes must be zero.

Conversely, if $\Sigma X = 0$ *and* $\Sigma Y = 0$, *the forces are in equilibrium and their resultant is zero.*

The foregoing law can be illustrated in another way. A fundamental principle of mechanics is that a body cannot be in equilibrium when acted upon by a single force. Now ΣX is the sum of all the components of the acting forces along any arbitrarily chosen axis. If a body is in equilibrium, there can be no change in motion. Therefore, there cannot be a force acting in any direction, or the body would move or change its motion.

Consequently, $\Sigma X = 0$, and $\Sigma Y = 0$.

The foregoing theorem can also be illustrated by the force polygon shown in Fig. 111. Suppose *ABCDEA* is the force polygon of five forces in equilibrium. Choose the axes *ox* and *oy*. Now the *x* component of a force is its projection on the *x* axis. The *x* projections are arranged as follows:

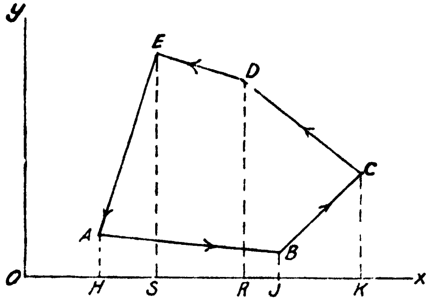


FIG. 111.

x projection of $AB = HJ$
x projection of $BC = JK$
x projection of $CD = KR$
x projection of $DE = RS$
x projection of $EA = SH$

But

$$\Sigma X = HJ + JK + KR + RS + SH = 0$$

That is, the sum of the positive projections is equal and opposite to the sum of the negative projections

A corollary to the preceding theorem is that, when the resultant effect of concurrent forces along any direction or axis is desired, simply project each of the forces onto the axis and find the algebraic sum of these projections.

70. Moments.—Figure 112 represents a capstan with a drum *AB* about which a rope is wrapped. By means of the rope a constant force *F* is being exerted on some load which is being moved. A force of 50 lb. applied at the end of the lever *CD* is just sufficient to keep the barrel turning at a constant speed. The distance from *E*, the center of the barrel to *D* is 10 ft. Then 50 lb. × 10 ft. = 500 ft.-lb. is a measure of the effort to turn the capstan. Experiment would show that if a force were applied at *K*, so that *EK* equals 8 ft., then the necessary force to apply at *K* is 62.5 lb., so that 62.5 × 8 = 500 ft.-lb. Also, 100 lb. applied at *H* 5 ft. out would give 100 × 5 = 500 ft.-lb., etc. Then

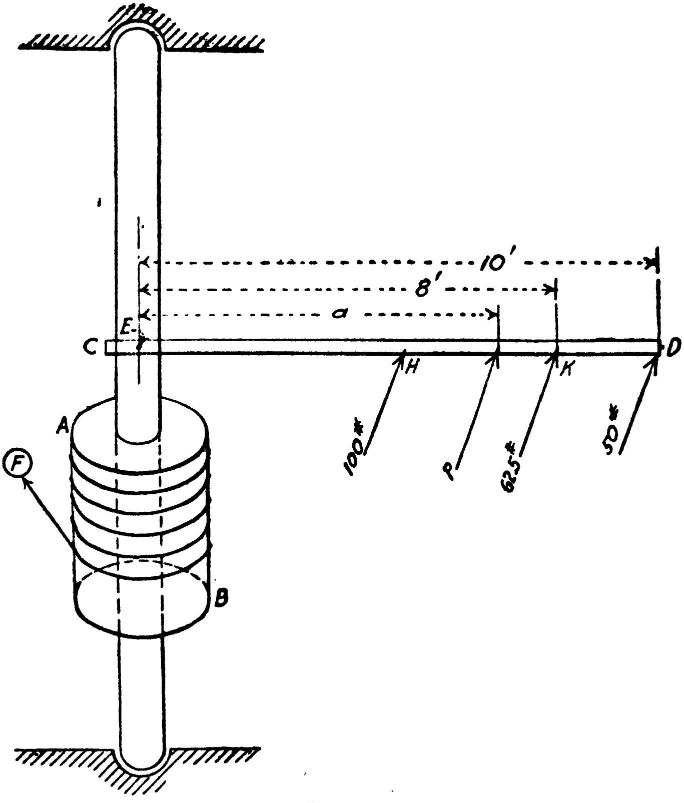


Fig. 112.

500 ft.-lb. is the measure of the effort necessary to move the given load and is called *the moment of the force about the vertical axis through E*. It is thus observed that the effect of the force varies with its distance from the axis. Such effect is measured by the product of the force and the perpendicular distance from the axis to the force.

The distance from E to the force is the perpendicular distance from E to the line of action of the force.

The *moment of a force about an axis* is defined as the force in pounds multiplied by the perpendicular distance in feet or inches from the axis to the force. For simplicity, the moment of a force about a point in a plane is to be understood as a moment about an axis perpendicular to the plane. Moment is expressed in foot-pounds or inch-pounds.

If P is the force and a is the distance, then the moment of P about $E = P \times a$ ft.-lb.

If the radius of the barrel is 1 ft. and the friction of the capstan in the bearings is neglected, the force F that would be exerted at the end of the rope is 500 lb., for the turning effort on the rope is equal to the turning effort of the lever, or

$$1 \times F = 50 \times 10 = 62.5 \times 8, \text{ etc.}$$
$$F = 500 \text{ lb.}$$

As a further illustration, if a force is applied to the rim of a pulley mounted on a shaft, the pulley will rotate. Experience shows that it is easier to turn the pulley when the force is applied to the rim than when it is applied closer to the shaft.

It is quite apparent that any effort to turn a wheel will not be applied parallel to a spoke but rather perpendicular to a spoke. Likewise, a force applied parallel to the axle will not rotate the wheel. From such observations, it is seen that no rotation or moment results when the line of action of the force either intersects the axis or is parallel to it. These several conditions will be found useful in deciding about what axis moments may be taken.

The term *moment of a force about a point* is used to express the product of a force times the perpendicular distance from the point, although there may not be a body at the point to rotate. Thus, in Fig. 113, the moment of P about O is

$$P \times OR$$

The following procedure will help to clarify the principle: Pass a plane through the force *P* and the point *O* about which moments are to be taken. Through *O*, imagine an axis *AB* drawn perpendicular to the plane *MN* fastened rigidly to *OR*. Then the effect of *P* is to rotate the imaginary axis *AB*.

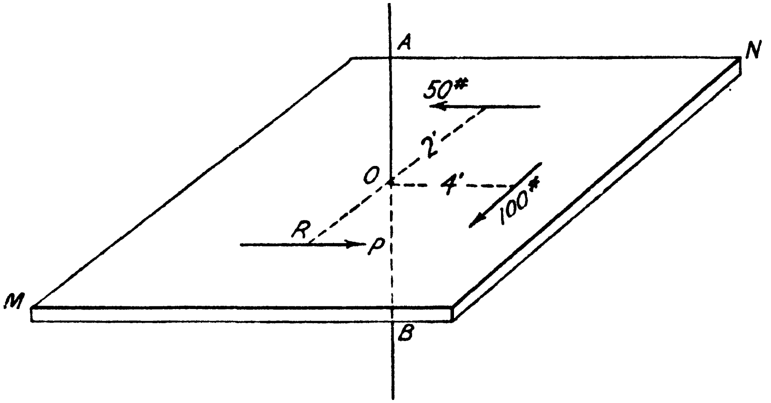


FIG. 113.

71. Sign of Moments.—In Fig. 113, it is evident that the 100-lb. force tends to rotate *AB* in one direction and the 50-lb. force tends to rotate *AB* in the other. Some rule for the direction of rotation is then necessary. When a force tends to produce clockwise rotation, the moment is negative. When the rotation is counterclockwise, the moment is positive. For example:

Moment 100 lb. $= -100 \times 4 = -400$ ft.-lb.
Moment 50 lb. $= +50 \times 2 = +100$ ft.-lb.

The total turning effect is $-400 + 100 = -300$ ft.-lb. (clockwise rotation).

72. The Moments of Concurrent Forces.—In Fig. 114, let F_1 and F_2 be force vectors and F their resultant. Assume any point *P*, and connect it with *O*; let distance *OP* be designated by *a*. Also θ_1, θ, and θ_2 will be the angles that

OP makes with the respective forces. Now taking moments about *P*, we have

$$\begin{aligned} \text{Moment of } F_1 &= F_1 a \sin \theta_1 \\ \text{Moment of } F_2 &= F_2 a \sin \theta_2 \\ \text{Moment of } F_1 + \text{moment of } F_2 &= F_1 a \sin \theta_1 + F_2 a \sin \theta_2 \\ &= a(F_1 \sin \theta_1 + F_2 \sin \theta_2) \end{aligned}$$

But, from the geometry of the figure, by resolving forces perpendicular to *OP* it can be seen that

$$F \sin \theta = F_1 \sin \theta_1 + F_2 \sin \theta_2$$

And

$$F_1 a \sin \theta_1 + F_2 a \sin \theta_2 = Fa \sin \theta \tag{24}$$

Since the last term is the moment of the resultant with respect to point *P*, the equation states that the sum of the moments of the components is equal to the moment of the resultant with respect to any point that might be chosen. Since any number of forces may be reduced to two by successive combinations and treated as above, the following principle may be stated: *The sum of the moments of any number of concurrent forces with respect to a given point is equal to the moment of their resultant with respect to that same point.*

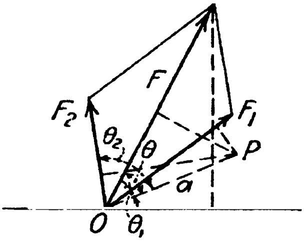


FIG. 114.

73. Moments and Equilibrium.—The principle of moments may be carried to a further conclusion for the condition of equilibrium. Since the resultant of the forces will be equal to zero, the moment of the resultant must be equal to zero. But, from the previous principle, the moment of the components is equal to the moment of the resultant. Therefore, when equilibrium exists, the moments of the forces involved are equal to zero. Since the moments of some of the forces must be clockwise and of others counterclockwise, it follows that in order to attain equilibrium the counterclockwise moments must be equal to the clockwise moments. This principle is expressed by the equation

$$\Sigma M = 0 \qquad (25)$$

Example 1.—A wheel 3 ft. in diameter and weighing 500 lb. is to be rolled over an obstruction 1 ft. high by a horizontal force P at the center of the wheel (Fig. 115*a*). Find the force P necessary to start the wheel and the resistance R of block A.

Solution.—The resultant of P and the weight of 500 lb. must be equal and opposite to the resistance to motion, if the wheel is just on the point of moving. The resisting force acts at A. The resultant of the other two forces acts through O. By the *principle of concurrence* the lines of action of all three forces must pass through O. Consequently, the direction of the resistance is AO. Set O out as a free body (Fig. 115*b*).

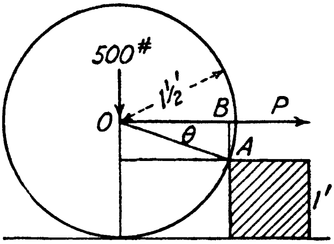


FIG. 115*a*.

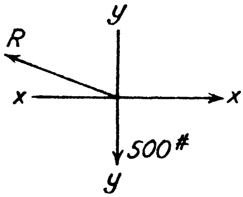


FIG. 115*b*.

From Fig. 115*a*,

$$OA = 1\tfrac{1}{2}, \qquad \text{and } AB = \tfrac{1}{2}$$
$$\therefore \sin\Theta = \tfrac{1}{3}, \qquad \text{and } \cos\Theta = 0.943$$

Now resolve each force into its components.

Force	x comp.	y comp.
P	P	0
500	0	−500
R	$-.943R$	$+\ \tfrac{1}{3}R$

Then

$$\Sigma X = P - 0.943R = 0$$
$$\Sigma Y = \tfrac{1}{3}R - 500 = 0$$

Solving the second equation, we have

$$R = 1{,}500 \text{ lb.}$$

From the first equation,

$$P = 0.943 \times 1{,}500 = 1{,}415 \text{ lb.}$$

Solution by Moments.—To find force P, take moments about A. Then

$$P \times \text{distance } BA = 500 \times \text{distance } OB$$

From the figure,

$$P \times 0.5 = 500 \times \sqrt{2}$$
$$P = \frac{500 \times \sqrt{2}}{0.5} = 1{,}415 \text{ lb. } \textit{Ans.}$$

Example 2.—Find the horizontal force F necessary to rotate the block about the point A, shown in Fig. 116. HINT.—The weight acts from C, the center of the block. The resisting force acts at A, the edge about which the block will turn. Also, the three forces F, 400 lb., and R at A must meet at a common point, for they are in equilibrium. That point is the center of the upper face BD.

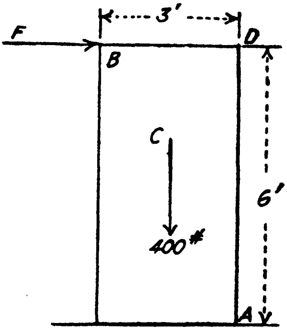


FIG. 116.

74. Design of Members in Simple Structures.—In the design of a simple structure like the derrick shown in Fig. 90, the stress in CB is found to be 4,333 lb., whereas in AB it was 1,666 lb. Since AB is a tensile member, it may be a steel rod or a cable, whereas the compression member CB may be a square timber either of oak or of southern pine. What must be their dimensions?

From Table I (Chap. I), a safe working stress for steel is 16,000 lb. per sq. in. Let A be the area of the rod. Then

$$16{,}000\,A = 1{,}666 \text{ lb.,} \qquad A = 0.104 \text{ sq. in.}$$

But

$$A = \frac{\pi d^2}{4}$$

$$\therefore \frac{\pi d^2}{4} = 0.104, \qquad d = 0.36 \text{ in.}$$

Any rod larger than 0.36 in. in diameter is perfectly safe A safe working compressive stress for southern pine is 1,200 lb. per sq. in. If b is the length of the side of the timber, $A = b^2$. Then

$$\begin{aligned} 1{,}200\, b^2 &= 4{,}333 \text{ lb.} \\ b^2 &= 3.61 \text{ sq. in.} \\ b &= 1.9 \text{ in.} \end{aligned}$$

Therefore, a timber 2 by 2 in. would safely sustain the load so far as compression is concerned. If the length were relatively great, the piece must be considered as a column, in which case failure might occur by buckling. The treatment of columns is found in a later chapter.

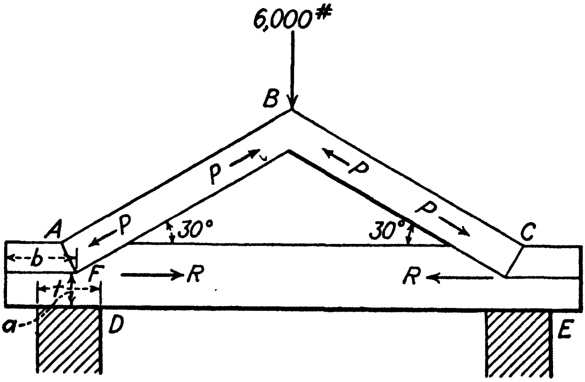


Fig. 117.

Example 1.—Figure 117 represents a simple roof truss. Members *AB* and *BC* of southern pine are pinned together at *B*. Find the necessary size of *AB* and *BC*. Find the length *b* to prevent shearing of the horizontal white oak beam at *A*. Also, find the width *t* of the support *D*.

Solution.—At the point *B*, there are three concurrent forces, the directions of which are known, since *AB* and *BC* are compression members. Set *B* out as a free body (Fig. 118). Then construct the triangle of forces. Since the force triangle is equilateral, $P = P =$ 6,000 lb. A safe working compression stress for southern pine is 1,200 lb. Then, if *A* is the area of the cross section of *AB*,

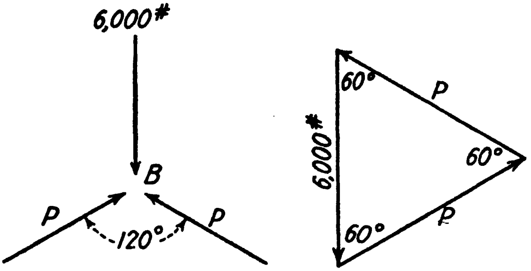


FIG. 118.

$$A = \frac{P}{1{,}200} = \frac{6{,}000}{1{,}200} = 5 \text{ sq. in.}$$

A 4- by 4-in. timber would be amply safe. A 3- by 3-in. timber would carry the load; but, for bearing area on *D* and for mortising at *A*, the 4- by 4-in. timber is chosen.

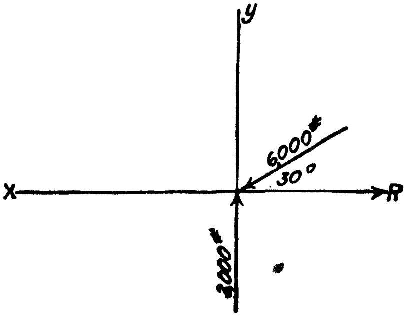


FIG. 118*a*.

At *A*, there are three concurrent forces: the resultant upward reaction at *D*, which on account of symmetry is one-half the load, or 3,000 lb.; the compression of 6,000 lb. in *AB*; and the unknown tension in *AC*.

The arrangement of the joint at *A* must be such that the center lines of *AB*, and of *AC* and the pier or abutment *D* meet in a point, as it is assumed that their resultant stresses act along their axes.

Set *A* out as a free body (Fig. 118*a*).

Choose the direction of R as the axis of X. Then

$$\Sigma X = R - 6{,}000 \cos 30^\circ = 0$$
$$R = 6{,}000 \times 0.866 = 5{,}196 \text{ lb.}$$

The shearing area is b in. long, 4 in. wide, or

$$A = 4b$$

A safe shearing stress in white oak is 200 lb. per sq. in. along the grain.

$$\therefore 200 \times 4b = 5{,}196 \text{ lb.}$$
$$\therefore b = \frac{5{,}196}{800} = 6.5 \text{ in. } \textit{Ans.}$$

The supports D and E are of southern pine. Since their dimensions are 4 in. and t in., the bearing area

$$A = 4t$$

If the grain is horizontal

$$s = 200 \text{ lb./sq. in.}$$
$$\therefore 200 \times 4t = 3{,}000 \text{ lb.} \qquad \therefore t = 3.75 \text{ in.}$$

If the grain is vertical,

$$s = 1{,}200 \text{ lb./sq. in.}$$
$$\therefore 1{,}200 \times 4t = 3{,}000 \text{ lb.} \qquad \therefore t = \tfrac{5}{8} \text{ in.}$$

Any width greater than the foregoing values of t would satisfy the requirements for safety.

The depth of AC must be sufficient to prevent vertical shearing at the lowest point F of the cut in the beam. Let a be the depth. Since AC is 4 in. wide, the vertical shearing area is $4a$. If the safe shearing stress across grain for white oak is 600 lb., then $600 \times 4a$ must equal the upward reaction of 3,000 lb.

$$600 \times 4a = 3{,}000 \text{ lb.} \qquad \therefore a = \tfrac{5}{4} = 1\tfrac{1}{4} \text{ in.}$$

Though F is not quite 4 in. below the top of AC, it is near enough to figure on a total depth of $4 + 1\frac{1}{4} = 5\frac{1}{4}$ in.

Then the nearest commercial size required would be a 4- by 6-in. white-oak beam.

Example 2.—In the preceding example, take a load of 8 tons, and design each member for the truss.

Example 3.—*BC* is a steel tie rod, and *AC* a white oak strut, in Prob. 98. Design these members. If *AC* is fastened to *DE* by a white oak pin at right angles to *AC*, find the necessary diameter to prevent the shearing of the pin.

Example 4.—Design the steel tie rod and southern pine compression member for Prob. 99.

75. Résumé of Methods.

In the foregoing pages a number of methods for the solution of problems have been developed. In order to call especial attention to them by way of review, these methods will be enumerated. Not all of them will be convenient for any one problem because the data may not be complete.

The methods are:

a. Graphical: measurement of force polygon
b. Trigonometric: solution of force polygon
c. Algebraic: by vertical and horizontal summation
d. Proportional: by comparison of force and space triangles
e. Geometric: polygon, made up of line segments
f. Moments

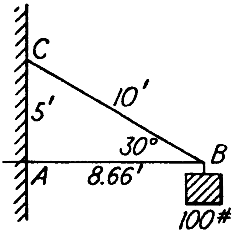


FIG. 119.

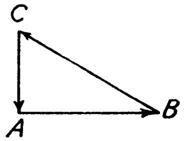


FIG. 119*a*.

Example.—Find the stresses in members *AB* and *BC* of Fig. 119, and illustrate the various methods.

Solutions.

a. Graphical; measurement of force polygon.

Draw the force triangle to scale as shown in Fig. 119*a*.

By measurement, the stresses are $AB = 173.2$, $BC = 200$ lb.

b. Trigonometric; solving for the sides of the triangle.

$$BC \sin 30° = 100, \qquad BC = \frac{100}{\sin 30°} = 200 \text{ lb.}$$

$$AB \tan 30° = 100, \qquad AB = \frac{100}{\tan 30°} = 173.2 \text{ lb.}$$

c. Algebraic:

Since $\Sigma Y = 0$,

$$BC \sin 30° - 100 = 0$$

$$BC = \frac{100}{\sin 30°} = 200 \text{ lb.}$$

Since $\Sigma X = 0$,

$$\begin{aligned} -BC \cos 30^\circ + AB &= 0 \\ -200 \times 0.866 + AB &= 0 \\ AB &= 173.2 \text{ lb.} \end{aligned}$$

d. Proportional:

It can be seen that the force triangle ABC is similar to the space triangle in Fig. 119. Therefore

$$\frac{\text{Stress in } BC}{10} = \frac{100}{5}, \qquad \text{or } BC = \frac{100 \times 10}{5} = 200 \text{ lb.}$$

Also,

$$\frac{\text{Stress in } AB}{8.66} = \frac{100}{5}, \qquad \text{or } AB = \frac{100 \times 8.66}{5} = 173.2 \text{ lb.}$$

e. Geometric and trigonometric methods are identical if the force polygon is a triangle.

f. Moments.

Selecting A as a center of moments, we then have

$$\begin{aligned} M_a &= 0 = -100 \times 8.66 + BC \times 8.66 \sin 30^\circ \\ BC &= \frac{100 \times 8.66}{8.66 \sin 30^\circ} = 200 \text{ lb.} \\ M_c &= 0 = -100 \times 8.66 + AB \times 5 \\ AB &= \frac{100 \times 8.66}{5} = 173.2 \text{ lb.} \end{aligned}$$

Problems

101. A body is acted on by an upward force of 12 lb. and a horizontal force to the right of 20 lb. What are the amount and the direction of the resultant?

102. Three forces of 40, 50, and 75 lb., respectively, make angles of 10, 30, and 120° with the x axis. What are the amount and the direction of the resultant?

103. If the forces in Prob. 102 are in equilibrium, what angles do they make with each other?

104. A body is supported by two cords each of which makes an angle of 50° with the vertical. The tensions in the cords are 65 lb. each. What is the weight of the body?

105. A simple truss supports a load of 4 tons (Fig. 120). Find the stresses.

106. A wheel 3 ft. in diameter weighs 2,500 lb. with its load (Fig. 121). Find the horizontal force necessary to start the wheel over an obstruction 6 in. high. Find the least force. (Forces act through the center.)

107. A wire 24 in. long will stand a straight pull of 100 lb. The ends are fastened to two points 21 in. apart on the same level. What weight suspended from the middle of the wire will break it?

108. The lifting force of a balloon is 600 lb. The anchor rope makes an angle of 65° with the vertical. Find the tension in the anchor rope and the horizontal pressure of the wind against the balloon.

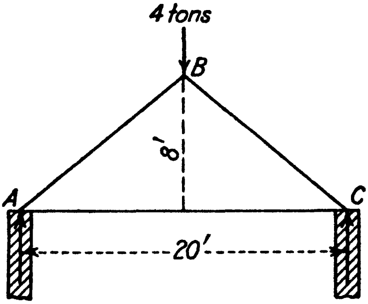


Fig. 120.

109. A truck weighing 2,000 lb. rests on a slope of inclination of 30°. Find what force will be needed to start the stalled truck up the grade if the resistance to traction is 50 lb.

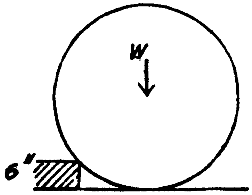


Fig. 121.

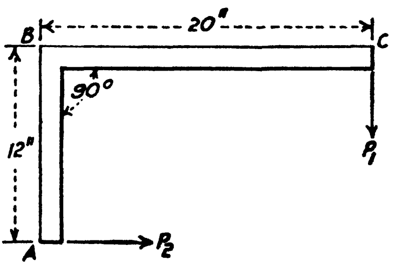


Fig. 122.

110. A bent lever *ABC* has its arms at right angles and is pinned at *B* (Fig. 122). *AB* is 12 in., *BC* is 20 in. Forces P_1 and P_2 are perpendicular to arms *BC* and *AB*, respectively. If $P_1 = 50$ lb., find P_2 and the magnitude and the direction of the pin reaction at *B*.

111. A horse is attached to a dump car by a cable which makes a horizontal angle of 10° with the track. It requires 200 lb. to move the car. Find the force that the horse exerts. What is the side pressure on the rails?

112. A truck bearing a load of 2 tons and standing on a smooth tramway of inclination 30° is connected by a chain to an empty truck on a plane of 45° and holds it at rest. If both trucks weigh the same when empty, find this weight.

113. A boiler weighing 2 tons is supported by cables making angles of 30 and 40°, respectively, with the vertical. Find the tension in each cable.

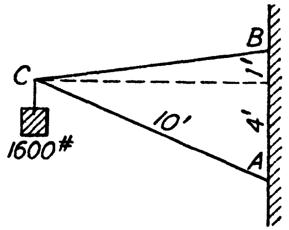


Fig. 123.

114. Find the stresses in *AC* and *BC* of Fig. 123.

115. A sailboat is being driven forward by a force of 400 lb. against the sail, as shown in Fig. 124. Find the force *P* that is actually driving the boat.

116. Find the stresses in *BC* and *AC* of Fig. 125. Use three methods.

117. If a force of 200 lb. is sufficient to move a body on a horizontal plane, find the force necessary to move it uniformly up a plane of slope 1 in 10, assuming the resistance the same in both cases.

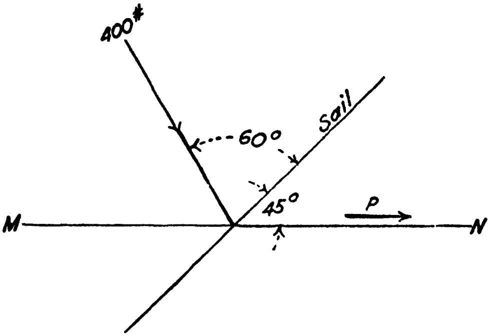


Fig. 124.

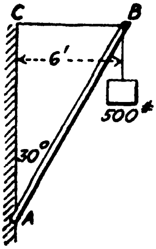


Fig. 125.

118. In Fig. 126, find the thrusts in *AB* and *BC*. Also, find the vertical and horizontal components of the reactions at *A* and *C*.

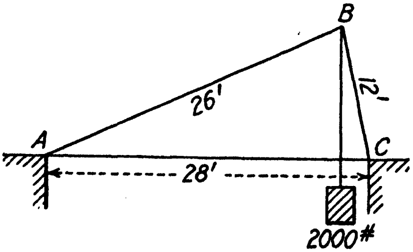


Fig. 126.

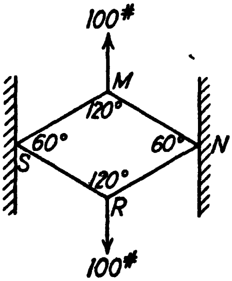


Fig. 127.

119. Figure 127 shows a frame made of four pieces of equal length with pivots at points *M*, *N*, *S*, and *R*. With the forces acting as shown, find the stresses in each of the arms and the horizontal reactions at S and N.

120. What stresses are produced in members AB and BC when the load is hoisted as in Fig. 128.

121. Given three forces of 50, 75, and 90 lb., making angles of 0, 70, and 120° with the x axis. Find the resultant.

122. If the forces of Prob. 121 act at such angles that they are in equilibrium, find the angles that they make with each other.

123. The pressure of the steam on the piston of a steam engine is 6,000 lb. Friction being neglected, what are the pressure on the guides and the stress in the connecting rod when the connecting rod makes an angle of 15° with the piston rod?

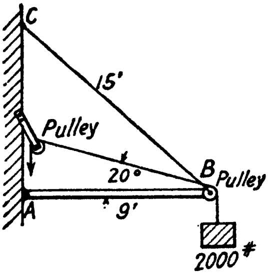


Fig. 128.

124. A boat is towed along a canal 30 ft. wide by mules on both banks. The length of each rope from its point of attachment to the bank is 80 ft. The boat moves down the middle of the canal. If the pull on each rope is 800 lb., what is the effective pull on the boat?

125. In Fig. 125, CB is an iron tie rod, and AB is a square oak strut. Find what dimensions of these members are necessary to carry the load safely.

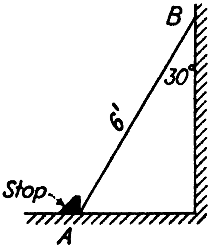


Fig. 129.

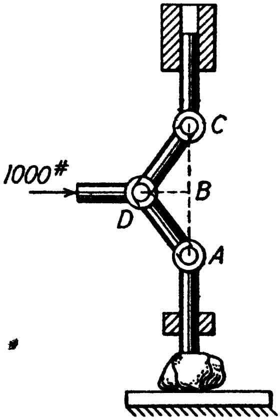


Fig. 130.

126. An iron bar 6 ft. long and weighing 200 lb. is leaning against a smooth vertical wall (Fig. 129). The bar makes an angle of 30° with the vertical and is prevented from sliding by a stop on the floor. What are the reactions at A and B?

127. In a toggle joint as shown in Fig. 130, what vertical pressure is exerted against the plunger due to the force of 1,000 lb.? $AB = 10$ in. $BD = 6$ in. Assume joint C to be stationary.

CHAPTER V

PARALLEL FORCES AND MOMENTS

76. Definitions and Illustrations.—When forces act in the same or in opposite directions, they are called *parallel forces.*

The principles involved in the study and application of parallel forces will be best understood by a simple experiment. Suspend a light rod BC (Fig. 131) by means of two spring balances at M and N, and note the readings. Now suspend 120 lb. from the point A. The point of the spring at M will move down 40 lb., and the one at N will move down 80 lb.

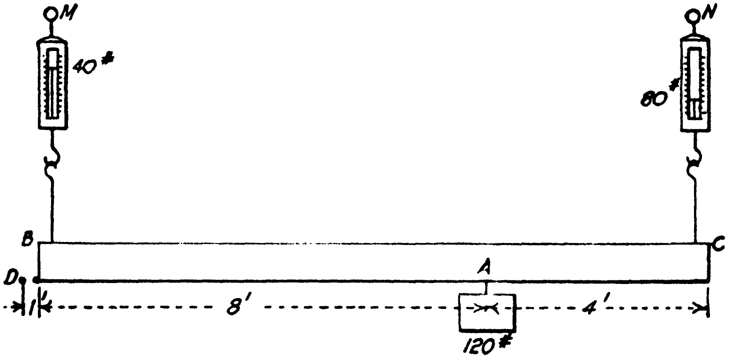


FIG. 131.

The forces at B, C, and A are parallel and in equilibrium. Their sum $40 + 80 - 120 = 0$.

Take moments of all the forces as follows:

First, moments about point B, represented by M_B.

$$M_B = -8 \times 120 + 12 \times 80 = -960 + 960 = 0$$
$$M_A = -8 \times 40 + 4 \times 80 = -320 + 320 = 0$$
$$M_C = -12 \times 40 + 4 \times 120 = -480 + 480 = 0$$
$$M_D = +40 \times 1 - 120 \times 9 + 80 \times 13 = 40 - 1{,}080 + 1{,}040 = 0$$

These results illustrate the principle that when parallel forces are in equilibrium, the sum of their moments about any point is zero.

Now parallel forces tend to *translate* and *rotate* the body on which they act. If such a body is in equilibrium, it must *neither translate nor rotate.*

The algebraic sum of the forces must be zero.

The algebraic sum of the moments about any point must equal zero.

In Chap. I, it was shown that if two forces are in equilibrium, they must be equal in magnitude and act through a common point.

The resultant of the 40 lb. at B and the 80 lb. at C must be their sum 80 + 40 = 120 and must act upward at the point A. Then

$$\frac{80}{40} = \frac{BA}{AC} = \frac{8}{4}$$

The resultant is applied at a point dividing the distance between the forces into two parts that are in an inverse ratio to the forces themselves.

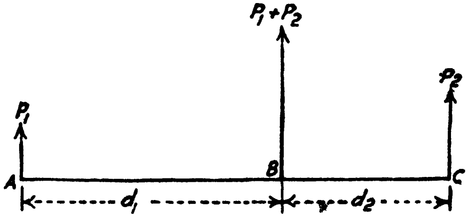


FIG. 132.

In Fig. 132, the resultant of P_1 and P_2 is their sum $P_1 + P_2$ and acts at B such that

$$\frac{P_1}{P_2} = \frac{d_2}{d_1} \tag{26}$$

Clearing of fractions produces

$$d_1 \times P_1 = d_2 \times P_2 \tag{A}$$

That is, *the moments of the two forces about a point on the resultant are equal to each other.*

Taking moments of the forces about A,

$$d_1(P_1 + P_2) = (d_1 + d_2)P_2$$

or

$$d_1P_1 + d_1P_2 = d_1P_2 + d_2P_2 \qquad \text{(B)}$$

The last equation could have been obtained by adding d_1P_2 to both sides of Eq. (A) above. The result shows that the resultant is placed so that its moment about a point on one of the forces is equal to the moment of the other force about the same point.

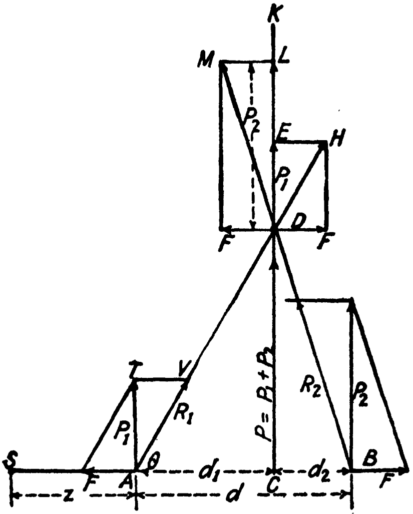


FIG. 133.

77. Resultant of Parallel Forces.—In Fig. 133, let P_1 and P_2 be two parallel forces so placed that AB through their ends is perpendicular to each. To prove the theorems stated in the preceding article, resolve P_1 into two components F and R_1 and P_2 into F and R_2. Since the two F forces neutralize each other, the resultant of P_1 and P_2 will be the resultant of R_1 and R_2. Extend them until they meet at D, and from this point lay off R_1 and R_2.

Now resolve R_1 into components, one equal and parallel to the original P_1 and the other equal to F. Triangles ATV and DEH are equal since $TV = EH, DH = AV$, and $\angle TAV = \angle EDH$. Therefore, DE is equal to and parallel to P_1. Similarly, R_2 is resolved into F and P_2 along DK. Since the F's at D neutralize each other, the resultant of R_1 and R_2 is the resultant of P_1 and P_2 acting at D along DK parallel to their original positions. Then $P = P_1 + P_2$. *The resultant of two parallel forces is equal to their sum and parallel to them.*

Now produce KD to intersect AB at C. Since triangles ACD and DEH are similar,

$$\frac{DC}{AC} = \frac{P_1}{F} = \frac{DC}{d_1} \tag{C}$$

Also,

$$\frac{DC}{CB} = \frac{P_2}{F} = \frac{DC}{d_2} \tag{D}$$

Dividing Eq. (C) by Eq. (D),

$$\frac{P_1}{P_2} = \frac{d_2}{d_1} \tag{26}$$

That is, *the resultant of two parallel forces passes through a point that divides the distance between the forces in the inverse ratio to the forces.* Clearing Eq. (26) of fractions, we have

$$P_1d_1 = P_2d_2 \tag{26a}$$

But these terms are the moments of the forces. Then the resultant is located so that the moment of P_1 about any point on P is equal to the moment of P_2 about the same point.

Adding P_2d_1 to both sides of Eq. (26a),

$$d_1(P_1 + P_2) = P_2(d_1 + d_2) \tag{E}$$

or

$$d_1P = dP_2 \tag{26b}$$

Therefore, the moment of P about any point on one of the forces is equal to the moment of the other force about the same point.

Now choose point S, at z units to the left of P_1, and multiply the equation $P = P_1 + P_2$ by z, then,

$$zP = zP_1 + zP_2 \tag{F}$$

Adding Eqs. (E) and (F),

$$(z + d_1)P = zP_1 + (z + d)P_2 \tag{26c}$$

The moment of the resultant of two parallel forces about any point is equal to the sum of the moments of the forces about the same point.

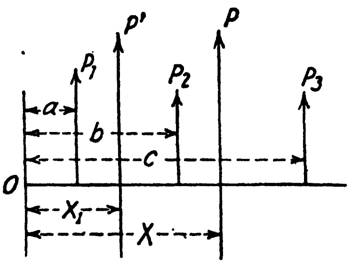


FIG. 134.

78. Resultant of More than Two Parallel Forces.—Suppose there are three parallel forces P_1, P_2, and P_3 (Fig. 134). Now P' the resultant of P_1 and P_2 is

$$P' = P_1 + P_2$$

and its distance x_1 from 0 is found from Art. 77, by the equation

$$P'x_1 = aP_1 + bP_2 \tag{26c}$$

The resultant P of P' and P_3 is

$$P = P' + P_3 = P_1 + P_2 + P_3 \tag{G}$$

The position of P is found from the moment equation, *viz.*,

$$Px = x_1P' + cP_3$$

$$Px = aP_1 + bP_2 + cP_3 \tag{H}$$

$$\text{or } x = \frac{aP_1 + bP_2 + cP_3}{P_1 + P_2 + P_3} \tag{I}$$

By adding still more parallel forces and continuing the same line of reasoning, the following theorem may be stated: *The resultant of any number of parallel forces is* *parallel to these forces and equal to their sum, and its moment about any point is equal to the sum of the moments of all the forces about the same point.*

Although in the preceding demonstration all the forces are assumed in the same direction, the theorem is true if some of the forces are in the opposite direction.

If a set of parallel forces is in equilibrium, then $P = 0$, and its moment $Px = 0$. From Eqs. (G) and (H) above,

$$P_1 + P_2 + P_3 + \cdots = 0$$

and

$$aP_1 + bP_2 + cP_3 + \cdots = 0$$

Example 1.—A bar 10 ft. long carries a weight of 20 lb., 6 ft. from the end. What force must be applied at each *end* to support the rod (Fig. 135)?

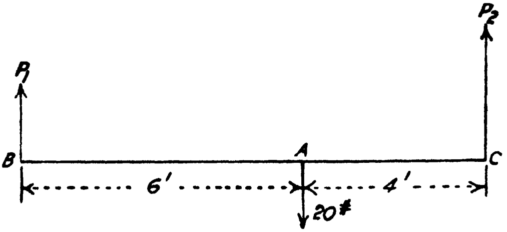


FIG. 135.

Solution.—The resultant of P_1 and P_2 must be 20 lb. applied at A. By the inverse ratio,

$$\frac{P_1}{P_2} = \frac{4}{6}$$

That is, $\frac{4}{10}$ of the load is at B, or $\frac{4}{10} \times 20 = 8$ lb. Also, $\frac{6}{10}$ is at C, or $\frac{6}{10} \times 20 = 12$.
By the principle of moments,

$$M_B = -6 \times 20 + 10P_2 = 0$$

since there is equilibrium.

$$\therefore P_2 = 12 \text{ lb.}$$

Since $P_1 + P_2 = 20$,

$$P_1 = 20 - 12 = 8 \text{ lb.}$$

If two parallel forces act opposite to each other, such as P_1 *and* P_2 *of Fig.* 136, *their resultant is equal to their difference, or* $P_1 - P_2$. The point of application of the resultant cannot be between the forces, for thus they would both be rotating in the same direction and could not balance. The point must be outside, and on the side of the largest force, so that the moment arm of the smallest force may be the greatest. For example, A must be situated so that

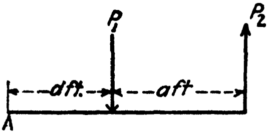


FIG. 136.

$$P_1 \times d = P_2(a + d)$$

Example 2.—Let $P_1 = 30$ lb., $P_2 = 15$ lb., and $a = 6$ ft. Then,

$$\begin{aligned} 30 \times d &= 15(d + 6) \\ 30d &= 15d + 90 \\ d &= 6 \text{ ft.} \end{aligned}$$

Example 3.—A beam resting on two end supports carries concentrated loads as shown in Fig. 137. Find the reactions of the supports.

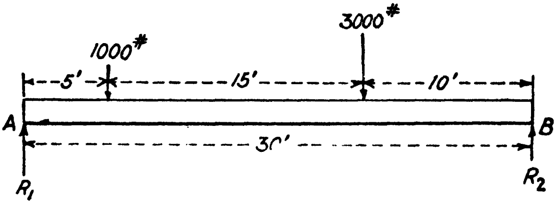


FIG. 137.

Solution.—Let R_1 and R_2 be the reactions. By the principle of moments, the algebraic sum of the moments about any point is zero. A convenient point to choose would be either the right or left support, for thus one of the unknowns would be eliminated. If the point A is chosen,

$$\begin{aligned} M_A &= -1{,}000 \times 5 - 3{,}000 \times 20 + 30R_2 = 0 \\ R_2 &= 2{,}167 \text{ lb.} \end{aligned}$$

But

$$\begin{aligned} R_1 + R_2 &= 1{,}000 + 3{,}000 = 4{,}000 \text{ lb.} \\ \therefore R_1 &= 4{,}000 - 2{,}167 = 1{,}833 \text{ lb.} \end{aligned}$$

The problem may also be solved by the inverse ratio. The two components of 1,000 lb., acting down at A and B, are found as follows.

$$\frac{\text{Component at } A}{\text{Component at } B} = \frac{25}{5} = \frac{5}{1}$$

Then

$$\begin{aligned} \text{Component at } A &= \tfrac{5}{6} \times 1{,}000 = 833 \text{ lb.} \\ \text{Component at } B &= \tfrac{1}{6} \times 1{,}000 = 167 \text{ lb.} \end{aligned}$$

Similarly,

$$\frac{\text{Component of 3,000 at } A}{\text{Component of 3,000 at } B} = \frac{10}{20} = \frac{1}{2}$$

Then

$$\begin{aligned} \text{Component of 3,000 at } A &= \tfrac{1}{3} \times 3{,}000 = 1{,}000 \text{ lb.} \\ \text{Component of 3,000 at } B &= \tfrac{2}{3} \times 3{,}000 = 2{,}000 \text{ lb.} \\ \text{Total force at } A &= 833 + 1{,}000 = 1{,}833 \text{ lb.} \\ \text{Total force at } B &= 167 + 2{,}000 = 2{,}167 \text{ lb.} \end{aligned}$$

But R_1 and R_2 are the equilibrants or balancing forces.

$$\therefore R_1 = 1{,}833, \qquad R_2 = 2{,}167$$

The resultant of several parallel forces may be found by determining the magnitude and position of the balancing force. The resultant is equal in magnitude but opposite in direction, at the point determined.

When a load is uniformly distributed over a beam, the total uniform load may be considered as acting at the center of the length of beam over which it is distributed, when the moment of such a load is desired.

The weight of a beam is a uniformly distributed load.

Example 4.—Find the reactions of the beam loaded as shown in Fig. 138 when the uniform load is 80 lb. per ft. of length.

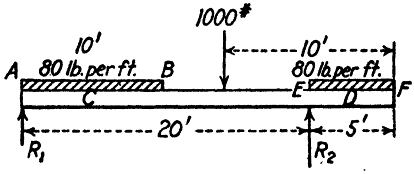


FIG. 138.

Solution.—The uniform load on AB is $10 \times 80 = 800$ lb. applied at C, 5 ft. from A. That on EF is $5 \times 80 = 400$ lb. applied at D, $2\tfrac{1}{2}$ ft. from E. By moments,

$$M_A = -800 \times 5 - 1{,}000 \times 15 - 400 \times 22.5 + 20R_2 = 0$$
$$\therefore R_2 = 1{,}400 \text{ lb.}$$
$$R_1 + R_2 = 800 + 1{,}000 + 400 = 2{,}200$$
$$\therefore R_1 = 2{,}200 - 1{,}400 = 800 \text{ lb.}$$

Problems

128. A beam 12 ft. long is supported at the ends. A load of 600 lb. is placed at 5 ft. from the left end. What are the reactions?

129. What should be the values of the forces R and F for the piece shown in Fig. 139?

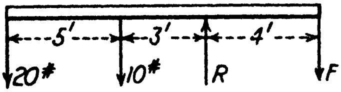


Fig. 139.

130. With the loads as shown in Fig. 140, determine the values of R_1 and R_2 by moments and also by the inverse-ratio principle.

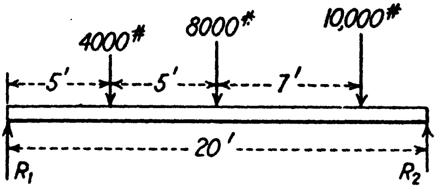


Fig. 140.

131. Figure 141 shows a beam weighing 40 lb. per lin. ft. This beam carries a load of 2,000 lb. 4 ft. from the right end and an additional load, uniformly distributed, of 300 lb. per ft. on the 6 ft. adjacent to the left end. Find R_1 and R_2.

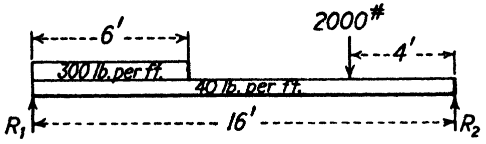


Fig. 141.

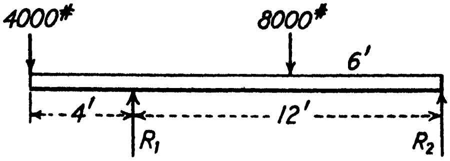


Fig. 142.

132. In Fig. 142, the beam overhangs the left support and is supported at the right end. The beam weighs 75 lb. per ft. With the other loads as shown, what are the reactions?

133. What are the reactions of the beam shown in Fig. 143? The beam weighs 30 lb. per lin. ft. Additional loads are indicated.

134. A beam supported at the ends is 20 ft. long and weighs 50 lb. per lin. ft. It carries a concentrated load of 1,800 lb., 9 ft. from the right end, and a total uniformly distributed load of 3,200 lb. What are the reactions?

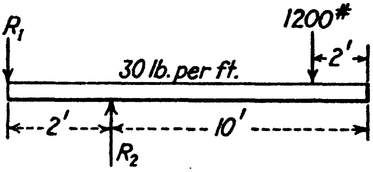


FIG. 143.

135. What force P is necessary to maintain equilibrium in the arrangement of the beams shown in Fig. 144? CD weighs 30 lb.; AB weighs 24 lb. and is pivoted at B.

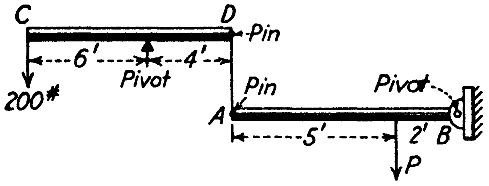


FIG. 144.

136. What are the end reactions for floor beams 16 ft. long and spaced 4 ft. from center to center? The total load on the floor including its own weight is 140 lb. per sq. ft.

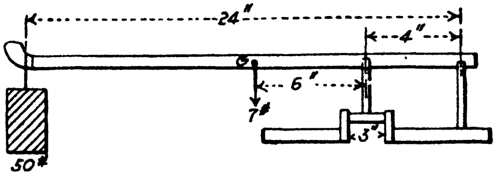


FIG. 145.

137. A safety valve shown in Fig. 145 is 3 in. in diameter and is just on the point of blowing off steam. The lever weighs 7 lb., and its center of weight is at G. The weight of the valve is 3 lb. If the weight on the end of the lever is 50 lb., find the pressure of the steam in the boiler. If the steam is to blow off at 80 lb., find the amount of the weight on the end of the lever.

138. Figure 146 shows an air cylinder and a system of levers. If the air pressure in the cylinder is 50 lb. per sq. in., what is the force exerted at F? If a force of 9,000 lb. is desired, what should be the air pressure?

139. Figure 147 represents a brake cylinder. The lever is bent at an angle of 120°. What force is produced in the brake rod by a pressure of 60 lb. per sq. in. in the cylinder?

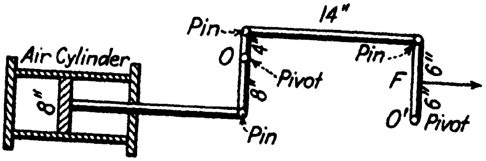


FIG. 146.

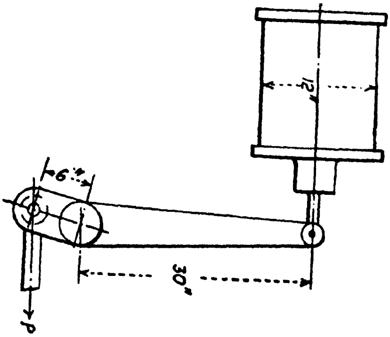


FIG. 147.

140. A pair of wire cutters is shown in Fig. 148. If the pressure is 15 lb. as shown, find the cutting force on the wire.

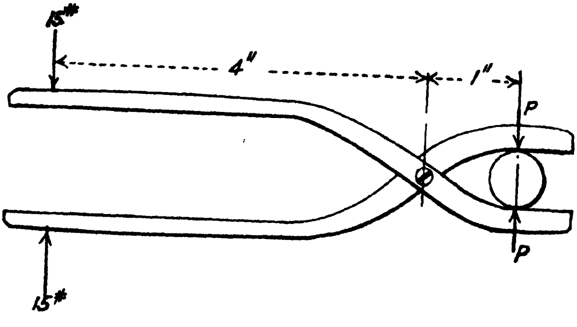


FIG. 148.

141. A throttle lever is shown in Fig. 149. What force will be available to move the throttle?

142. An engine reverse lever in shown in Fig. 150. What is the amount of the force?

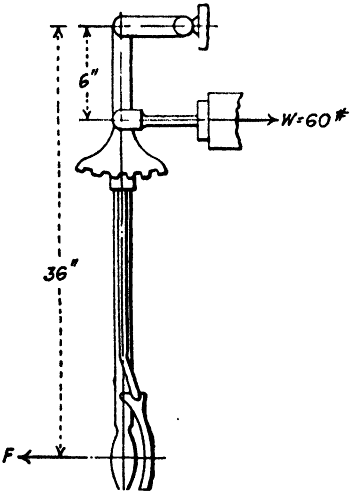


FIG. 149.

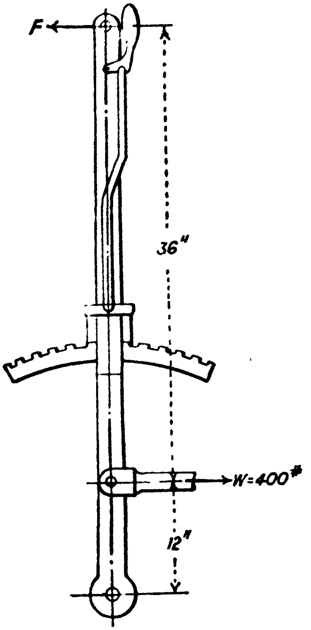


FIG. 150.

143. In Fig. 151, if P is 50 lb. and the weight is 1,000 lb., find the length BC so that the bar AC is in equilibrium. AB is 1 ft. long.

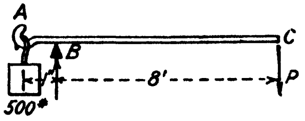


FIG. 151.

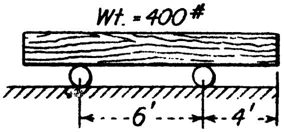


FIG. 152.

144. A timber of uniform weight and cross section is shown in position on the rollers while being moved (Fig. 152). What are the roller reactions? With the rollers still 6 ft. apart, what should be the position of the timber so that one roller will support twice as much as the other? The timber is 12 ft. long.

145. Find the position of the resultant load for a truck with a wheel base of 140 in., if the distribution of the load is such that the pressure on the rear axle is 3.5 tons and the pressure on the front axle is 2.5 tons.

79. Couple.—Figure 153 shows a boring tool. The hands are placed at A and C. One hand exerts a push of 20 lb., and the other a pull of 20 lb.

The forces are parallel, equal in magnitude, but opposite in direction and not along the same line of action. *Two such forces form a couple. A couple causes rotation only. The distance AC between the forces is called the "arm" of the couple.* The resultant R of the forces is $20 - 20 = 0$.

The moment, or turning effort, of the couple about B is the sum of the moments of the forces. both of which are of the same sign, for they turn in the same direction. Then

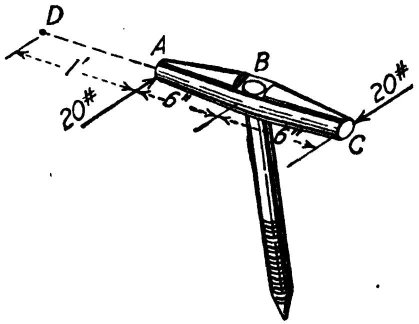


FIG. 153.

$$M_B = -20 \times \tfrac{1}{2} - 20 \times \tfrac{1}{2} = -20 \text{ ft.-lb.}$$

Moments about A give

$$M_A = -20 \times 1 = -20 \text{ ft.-lb.}$$

Choose a point D, 1 ft. to the left of A as a moment center. Then

$$M_D = 20 \times 1 - 20 \times 2 = -20 \text{ ft.-lb.}$$

The example *shows that the moment of this couple is a constant and is equal to one of the forces, times the arm, or distance between the forces.* The theorem is true in general, and, if the forces are F and F with an arm a, the moment is Fa.

Since a single force produces translation and a couple produces rotation, *a force cannot balance a couple.*

In Fig. 154, a single force P acts at a point D. At A, b ft. from D, draw two equal and opposite forces P. They neutralize each other, and the resultant effect remains the same. Now, forces along AB and DE form a couple with moment bP, tending to rotate the body about A. There remains the force P along AC, tending to translate the body.

A single force acting at a point, not the center of the body, causes rotation around the center and a translation of the center.

A single force acting at one point is equivalent to the same force, acting at a different point, plus a couple.

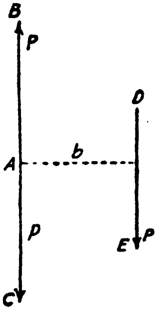


Fig. 154.

Common illustrations of the couple are seen in the forces that are applied to the steering wheel of the automobile or to valve wheels in waterworks or in hydraulic plants.

80. Practical Applications.—In many practical problems, there are equal and opposite forces discovered as the analysis proceeds, and the moment effect of these forces can at once be computed as a couple. Likewise, in certain problems the procedure of replacing a force acting at one point by means of a force at some other point plus a couple makes possible a definite solution. For instance, a hook supporting a load as shown in Fig. 155 presents such a case. The question to be answered is: What is the exact effect on the hook? It can be seen that the load would tend to straighten the curve ABC, but there is still much doubt in regard to the actual stresses produced. However,

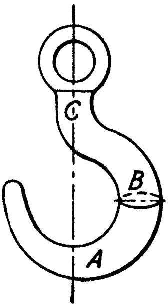


Fig. 155.

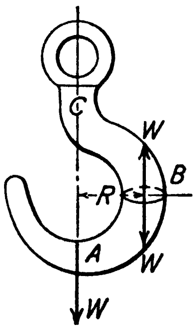


Fig. 155a.

if at section B in Fig. 155a the equal and opposite forces W are added as shown, the new system is equivalent to the old, for the same amount has been added as has been subtracted. The system in Fig. 155a consists essentially of a direct pull at B, causing a tensile stress, plus a couple consisting of the force W at A and the upward force W at B. This couple tends to rotate the part AB about B and produces a bending stress which can now be very definitely found, as shown later in Chap. XII.

Problems

146. A load of 100 lb. is applied at a point 4 ft. from the middle of a beam. Resolve the load into a force at the middle and a couple.

147. Forces of 10 lb. each are applied on opposite sides of a steering wheel 18 in. in diameter to form a couple. What is the moment of the couple? If a single force of 20 lb. is applied to the rim of the steering wheel, how is a couple produced and what is its moment?

148. In Prob. 128, show the forces as couples.

149. In Prob. 130, rearrange the forces as couples.

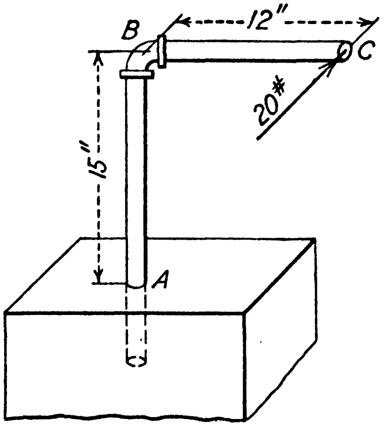


FIG. 156.

150. An iron pipe 1 in. in diameter is embedded in a concrete base (Fig. 156). A force of 20 lb. is applied perpendicular to BC. Analyze the effect of this force on the several parts of the pipe.

CHAPTER VI

STATIC AND KINETIC FRICTION

81. Maximum Friction. *Coefficient of Friction.*—Friction was defined in Art. 65 as a force acting between two surfaces in contact to resist the motion of one surface over the other. Friction is a passive force. It acts only when there is an effort made to move one body over the other.

In Fig. 157, let a 100-lb. body rest on a rough plane, and be acted on by a horizontal force P. If the weight does not move, the forces are in equilibrium, then

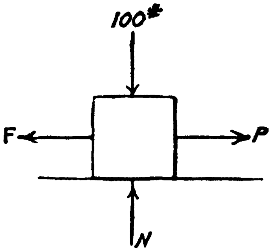


FIG. 157.

$$P = F, \qquad \text{and } N = 100$$

i.e., when $P = 10$ lb., $F = 10$ lb.; $P = 20$ lb., $F = 20$ lb.; etc.

But if $P = 30$ lb. and the body is just on the point of moving, $F = 30$ lb. is its *greatest,* or *maximum, static* friction. Any value of P, say 31 lb. or more will cause motion.

Experiment has shown that within certain limits the maximum value of friction depends on the normal pressure on the plane and on the materials in the two rubbing surfaces. *So long as the normal pressure per square inch does not exceed the elastic limit of the bearing surface, the friction is independent of the area of contact between the two surfaces but does depend on the normal pressure between the surfaces.*

Experiment has shown that for practical purposes the ratio of the maximum friction to the normal pressure on the plane is a constant, for any given material. *This ratio* *is defined as the "coefficient of static friction."* Let f represent the coefficient of friction and F_1 the maximum friction. Then

$$f = \frac{F_1}{N} \tag{27}$$

For Fig. 157,

$$f = \tfrac{30}{100} = 0.3$$

The resultant of F_1 and N when motion is impending is a single force, as shown in Fig. 158. Since P, W, and R are in equilibrium, they form a triangle such as OAB (Fig. 159). Let I be the angle that R makes with the normal to the plane.

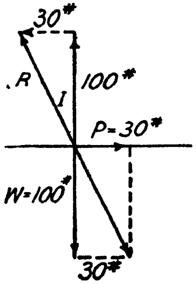


FIG. 158.

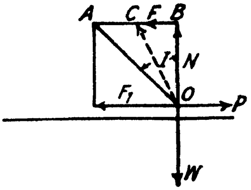


FIG. 159.

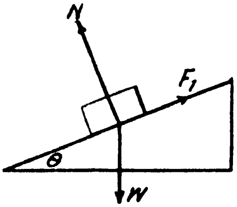


FIG. 160.

Then $\tan I = \tfrac{30}{100} = 0.3$; that is, when motion is impending, *the resultant reaction or resistance to motion makes an angle whose tangent is the coefficient of friction f. R is always drawn on the opposite side of the normal from the direction of motion.* That is, if motion would take place to the right, R bears to the left, etc. This principle is evident from the fact that F is always resisting motion.

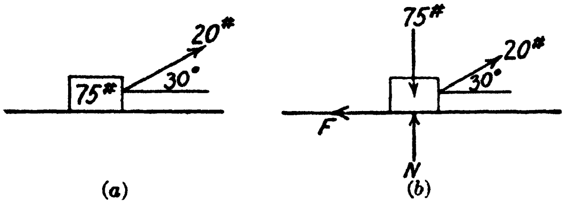
In Fig. 159, let $\angle AOB = I$ the maximum angle of friction. Suppose P is less than F_1; then the friction F is less than F_1, say BC. Then OC is the resultant resistance, making an angle COB with the normal. Motion cannot take place until P is greater than BA.

From Eq. (27), $F_1 = fN$.
If $f = 0.25$ and $N = 50$, $F_1 = 12.5$ lb.
If $f = 0.2$ and $N = 75$ lb., $F_1 = 15$ lb.

Every surface has depressions on its face. These depressions vary with the material and are more evident in a rough board than in one with a planed surface. When one body rests on another, the elevations on one tend to fill the depressions on the other. Before motion can ensue, one body must be lifted out of the elevations, or the surface leveled or smoothed. Although this illustration does not explain all the action of friction forces, it at least furnishes an explanation of why a body moves more easily over ice than a brick pavement, and also why it is easier to move a weight *by pulling at an angle to give a small vertical or lifting component.*

The force necessary to continue uniform motion is always less than the force necessary to start it. The friction of motion such as comes from revolving shafts, wheels on brake shoes, etc., is called *kinetic friction.*

Example 1.—A 75-lb. body rests on a horizontal plane for which $f = 0.4$. The body is acted on by a force of 20 lb. at 30° with the horizontal, as shown in Fig. 161*a*. Will the body move?



(*a*) (*b*)

FIG. 161.

Solution.—Set the weight out as a free body (Fig. 161*b*) These forces are in equilibrium vertically.

$$\therefore N - 75 + 20 \sin 30° = 0$$
$$N = 65 \text{ lb.}$$

By Eq. (27), maximum friction

$$F_1 = 0.4N = 0.4 \times 65 = 26 \text{ lb.}$$

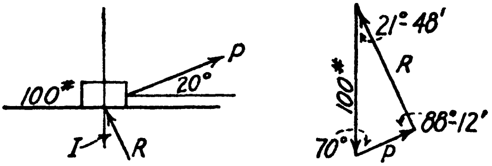
For equilibrium horizontally,

$$-F + 20 \cos 30° = 0$$
$$F = 17.32 \text{ lb.}$$

But the body will not move until maximum friction is overcome. Since 17.32 lb. is all the force necessary to prevent motion, the body remains at rest. Friction equals 17.32 lb.

Example 2.—A 100-lb. weight rests on a horizontal plane for which $f = 0.4$. A force P acts on the body at an angle of 20° with the horizontal. Find its magnitude for impending motion (Fig. 162*a*).

Solution.—When motion is impending, it will often be found convenient to use the resultant R of N and F_1, which must make the $\angle I = \tan^{-1} (f)$ with the normal. Setting out the free body Fig. 162*a*, we find there are three forces in equilibrium that may be taken as the sides of a triangle. R is drawn at the $\angle \tan^{-1} (0.4) = 21°48'$ with the normal, on the side to oppose motion. Figure 162*b* shows the force triangle. By the law of sines,



(*a*) (*b*)

Fig. 162.

$$\frac{P}{\sin 21°48'} = \frac{100}{\sin 88°12'}$$
$$P = \frac{100 \times 0.3714}{0.9995} = 37.14 \text{ lb.}$$

This problem may be solved by using the summation method. Since the components of the forces in the horizontal direction are in equilibrium at the instant of impending motion, from

$$\Sigma F_x = 0$$
$$P \cos 20° = 0.4(100 - P \sin 20°)$$
$$P \cos 20° + 0.4P \sin 20° = 0.4 \times 100$$
$$P = \frac{40}{(0.9397 + 0.1368)} = 37.14 \text{ lb. } \textit{Ans.}$$

It can be seen that the force P exerts a lifting component and reduces the amount of the normal pressure.

TABLE OF COEFFICIENTS OF STATIC FRICTION

Kind of Surfaces	f
Metal on metal	0.15–0.35
Wood on metal	0.20–0.60
Wood on wood	0.30–0.70
Leather on metal	0.30–0.60
Leather on wood	0.25–0.50
Stone on metal	0.30–0.40
Stone on stone	0.40–0.70

Above are given a few values of coefficients of friction. The degree of smoothness of the surfaces affects the value of f very materially, and this fact accounts for the range of values. The kinetic coefficients are always from 20 to 40 per cent lower than the values of static f.

82. Friction on an Inclined Plane.

82. Friction on an Inclined Plane.—Let a body weighing W lb. rest on a rough inclined plane. Suppose the weight is just on the point of slipping down the plane. Then friction is a maximum F_1. From Art. 65, the component of W down the plane is $W \sin \Theta$, and that normal to the plane is $W \cos \Theta$.

$$\therefore F_1 = W \sin \Theta, \qquad N = W \cos \Theta$$

By division,

$$\frac{F_1}{N} = \frac{W \sin \Theta}{W \cos \Theta} = \tan \Theta$$

But $F_1/N = f = \tan I$, where I is the angle of friction, or the angle that the resultant resistance makes with the normal.

Then $\tan I = \tan \Theta$, $I = \Theta$; *i.e., when a body rests on an inclined plane, if the angle of friction I is less than Θ, the angle of the plane, the body will slip. If $I > \Theta$, the body will not slide down the plane.*

Example 1.—A body weighing 50 lb. rests on a 20° plane. $f = 0.2$. Will the body slide down the plane? What is the force of friction under the weight?

Solution.—$I = \tan^{-1} 0.2 = 11°20'$. Since $I < \Theta$, the body will slide.

Now

$$N = W \cos \Theta = 50 \cos 20^\circ = 47 \text{ lb.}$$
$$F = 0.2N = 0.2 \times 47 = 9.4 \text{ lb}$$

The component of W down the plane is

$$W \sin \Theta = 50 \sin 20^\circ = 17.1 \text{ lb.}$$

Since $17.1 > 9.4$, the body must move.

Example 2.—In the preceding example, let $f = 0.3$ and $\Theta = 15^\circ$. Will the body move? How large is friction?

Solution.—$I = \tan^{-1} 0.3 = 16^\circ 42'$.

Since $I > \Theta$, the body will not move.

Now

$$N = W \cos \Theta = 50 \cos 15^\circ = 48.3 \text{ lb.}$$
$$F_1 \text{ (max)} = 0.3 \times 48.3 = 14.49 \text{ lb.}$$

But the force trying to move the body down the plane is

$$W \sin \Theta = 50 \sin 15^\circ = 12.9 \text{ lb.}$$

Since 12.9 lb. < 14.49 lb., it is evident that the body will not move. But friction is a passive force. Only enough resistance is offered to prevent motion. Therefore, the friction force actually developed is 12.9.

$$\therefore \text{Friction developed} = 12.9 \text{ lb.}$$

83. Force Necessary to Move a Body on a Rough Plane.—Figure 163 shows a body on a plane with a force P acting on the body parallel to the plane. If motion is impending up the plane, R the resultant of F_1 and N makes an angle I (the $\tan^{-1} f$) with the normal to the plane and is drawn on the left-hand side. There are then three concurrent forces in equilibrium. Construct the triangle of forces by drawing AB equal to W the only known force. From B, draw a line parallel to P. Since the figure closes, R must

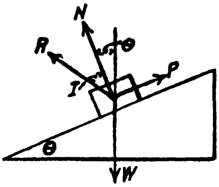


FIG. 163.

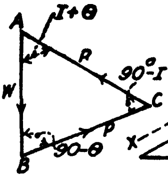


FIG. 163a.

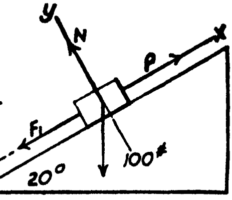


FIG. 163b.

end at A. Then, from A, draw a line parallel to R, forming the triangle ABC (Fig. 163*a*). Since the normal makes an angle Θ with the vertical, angle A of the triangle is $I + \Theta$.

$\angle B = 90 - \Theta$

$\therefore \angle C = 180 - (90 - \Theta + I + \Theta) = 90 - I$

This is easily seen from Fig. 163*a*.

By the law of sines, the force P necessary to just start the body up the plane is

$$\frac{P}{W} = \frac{\sin (I + \Theta)}{\sin (90 - I)}, \qquad P = W\frac{\sin (I + \Theta)}{\cos I}$$

Example 1.—Let $W = 100$ lb., $\Theta = 20°$, and $I = \tan^{-1} 0.3 = 16°42'$.
Solution.

$$P = \frac{100 \sin (20° + 16°42')}{\cos 16°42'} = \frac{100 \times 0.5972}{0.9580} = 62.4 \text{ lb.}$$

Another solution is as follows:
Choose axes as in Fig. 163*b*.

$$\begin{aligned} \Sigma X &= P - F_1 - W \sin 20° = 0 \\ \Sigma Y &= N - W \cos 20° = 0 \\ N &= 100 \times 0.94 = 94 \text{ lb.} \\ F_1 &= 0.3 \times 94 = 28.2 \text{ lb.} \end{aligned}$$

$\therefore P = 28.2 + 100 \sin 20 = 28.2 + 34.2 = 62.4$ lb. *Ans.*

Example 2.—Find the force P, parallel to the plane, just to start the body down the plane.

$$P = \frac{W \sin (\Theta - I)}{\cos I} \quad \textit{Ans.}$$

HINT.—Since motion is to be down the plane, friction acts up the plane. R makes an angle I on the right-hand side of the normal.

Example 3.—Find P in the preceding problem, if $W = 100$, $\Theta = 20°$, and $f = 0.3$; if $\Theta = 12°$ and $f = 0.3$.

NOTE.—If P is negative, it shows that a push is necessary to move the body down the plane. Evidently this would be the case if I is greater than Θ.

Example 4.—If, in Fig. 163, P is a horizontal force, prove that, for motion up the plane, $P = W \tan (\Theta + I)$.

For motion down the plane, $P = W \tan (\Theta - I)$.

84. Least Force.—Let a body W rest on a horizontal plane (Fig. 164). A force P acts at an angle Θ. Find the least force to just move the body. R, the resultant reaction, makes an angle $I = \tan^{-1} f$ with the normal. There are three concurrent forces in equilibrium. Draw AB equal to W. Since the direction of P is unknown, we draw a line parallel to R from A. The diagram must close, and P must be drawn from B to some point on AD and is to be the least force, represented by the shortest line. But the shortest line from B is a perpendicular to AD, *viz.*, BC.

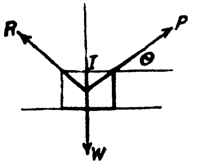


Fig. 164.

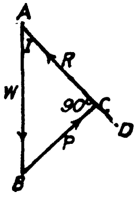


Fig. 164*a*.

Since P is perpendicular R, P makes an angle I, the angle of friction with the horizontal, the direction of impending motion. But ABC of Fig. 164*a* is a right triangle.

$$\therefore P = W \sin I$$

Example 1.—Let $W = 50$ lb. and $f = 0.3$. Find the horizontal force to just move the body; the least pull.

Solution.—If P is horizontal, $P = F$.

$$N = W, \quad f = 0.3, \quad F = 15 \text{ lb.}, \quad \therefore P = 15 \text{ lb.}$$

But, for least force,

$$P = W \sin I = 50 \sin 16°42' = 14.39 \text{ lb.}$$

Example 2.—Show that the least force P necessary to start a body up a plane, angle θ, with $I = \tan^{-1} f$, is given by the equation $P = W \sin (I + \theta)$.

This result shows that the least force is required when it acts at an angle I with the direction of impending motion.

Example 3.—Parcels are to move with uniform motion down a chute. If the coefficient of friction is 0.15, find the angle of inclination of the chute.

85. Cone of Friction.—If the line OA of Fig. 159 making the maximum angle θ of friction with the normal is revolved about OB as an axis, *the cone generated is called the "cone of friction."* If the resultant R of the normal reaction and the friction falls within the cone of friction, the forces acting on the body are not great enough to cause motion. This principle is used in self-locking mechanisms and also in cotter pins. For example, if the angle of the cotter pin is less than the angle of friction, no force at right angles to the axis of the pin could cause the bolt to move in the direction of its axis. This principle will be illustrated by applying it in the solution of some problems.

Example.—A lift slides on a vertical shaft 3 in. in diameter. Find the greatest distance from the shaft at which any load W can be placed, and the lift still slide on the shaft. $f = 0.3$. Disregard the weight of the lift (Fig. 165).

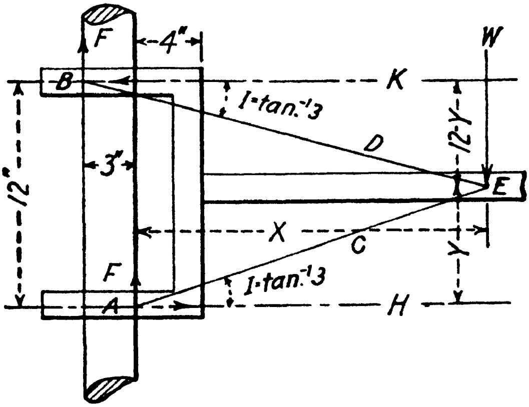


FIG. 165.

Solution.—When the load is placed on the lift, pressure is exerted perpendicular to the shaft, on the right side at the bottom and on the left side at the top. Let A and B be the points of application of these resultant horizontal pressures AH and BK. Friction at A and B is called F, in the direction indicated by vectors. Since motion is taking place, or impending, maximum friction is developed, and the resultant reaction at A is along AC at the angle $I = \tan^{-1}$ (0.3) with the normal AH. At B, it is along BD at the same angle with the horizontal.

For impending motion, the lift is in equilibrium under the action of three forces. They must meet in a point, *viz.*, point E, at which

AC and *BD* meet. From triangles *KBE* and *HAE*,

$$\tan I = \frac{Y}{X} = 0.3, \qquad \text{and} \qquad \frac{12 - Y}{X + 3} = 0.3$$
$$Y = 0.3X, \qquad 12 - Y = 0.3X + 0.9$$

By elimination,

$$\begin{aligned} 12 - 0.3X &= 0.3X + 0.9 \\ 0.6X &= 11.1 \\ X &= 18.5 \text{ in.} \end{aligned}$$

That is, with *W* less than 18.5 in. from the edge of the shaft, the lift will slide.

Solution by the Summation Method.—From

$$\Sigma Y = 0, \qquad (23)$$
$$F_A + F_B - W = 0 \qquad (23a)$$
$$\Sigma X = 0, \qquad (23)$$
$$N_A - N_B = 0$$
$$M = 0,$$
$$F_B \times 3 + WX - N_B \times 12 = 0 \qquad (25)$$

Since $N_A = N_B$,

$$F_A = F_B \qquad (27)$$

From (23*a*) above,

$$2F = W, \qquad \text{or} \qquad F = \frac{W}{2}$$

Since $F = fN$,

$$N = \frac{F}{f} = \frac{2W}{f}$$

Substituting in (25) above, we have

$$\frac{W}{2} \times 3 + WX - \frac{W}{2f} \times 12 = 0$$

Canceling *W* produces

$$\frac{3}{2} + X - \frac{12}{2 \times 0.3} = 0$$
$$X = \frac{6}{0.3} - 1.5 = 20 - 1.5 = 18.5 \text{ in.}$$

86. Wedge Action.—A wedge is a piece of wood or metal used in lifting heavy weights, splitting logs, etc. It may be made with one or two inclined faces. In Fig. 166 is shown a weight of 3,000 lb. to be lifted by means of two wedges with faces inclined at an angle of 15° with the horizontal. Coefficient of friction for all rubbing surfaces is 0.3. Find the forces F to just lift the weight.

First set the weight out as a free body (Fig. 166*a*). It is in equilibrium under the action of the 3,000-lb. weight and the two resultant reactions R_1 of the two rubbing surfaces. The direction of R_1 is determined by first drawing a line normal to each of the inclined surfaces at their centers. The motion of the weight relative to the wedge is upward. Therefore, the force R_1 must act inclined to the normal on the left side, as shown in the figure, to resist this motion. The angle that R_1 makes with the

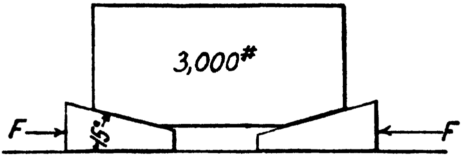


FIG. 166.

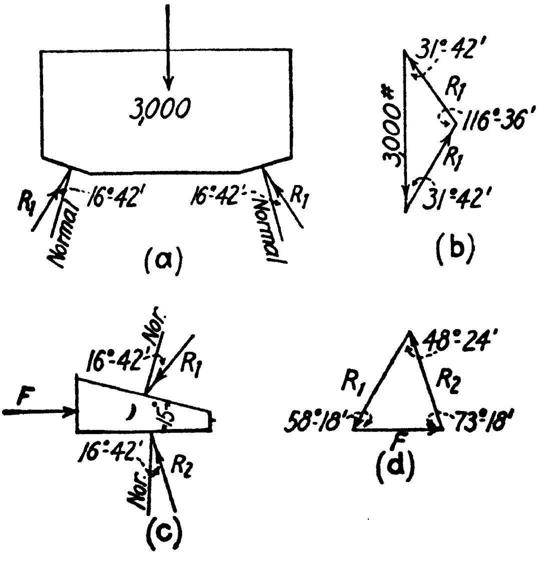


FIG. 166*a–d*.

normal is $I = \tan^{-1}(0.3) = 16°42'$. Now, draw the force triangle, Fig. 166*b*. By the law *of sines*,

$$\frac{R_1}{3{,}000} = \frac{\sin 31°42'}{\sin 116°36'}$$

$$R_1 = 1{,}760 \text{ lb.}$$

Now, set the wedge out as a free body (Fig. 166*c*). It is in equilibrium under the action of force F and the resultant reactions of the two rubbing surfaces. Since motion is impending, R_1 and R_2 must make the angle of friction with the normals at the centers of the rubbing surfaces. The motion of the wedge is to the right. This motion is resisted by the two surfaces in contact with the wedge. Therefore, R_1 and R_2 are drawn to the right of and at an angle of 16°42′ with their respective normals. Now, draw the force triangle (Fig. 166*d*).

By the law of sines,

$$\frac{F}{R_1} = \frac{\sin 48°24'}{\sin 73°18'}$$

$$F = 1{,}374 \text{ lb.}$$

The two free-body diagrams show that R_1 in one figure is equal, parallel, and opposite to R_1 in the other. This fact is easily understood because action is equal and opposite to reaction.

For this type of problem the solution by means of the force triangle is probably the simplest. For purposes of comparison, the summation method will now be applied. For the body,

$$\Sigma Y = 2R_1 \cos 31°42' - 3{,}000 = 0$$

$$R_1 = \frac{1{,}500}{\cos 31°42'} = 1{,}760 \text{ lb.}$$

For the wedge,

$$\Sigma Y = R \cos 16°42' - 1{,}760 \cos 31°42' = 0$$

$$R_2 = \frac{1{,}760 \times 0.8508}{0.9578} = 1{,}563.3 \text{ lb.}$$

$$\Sigma X = F - 1{,}760 \sin 31°42' - 1{,}563.3 \sin 16°42' = 0$$

$$F = 1{,}760 \times 0.5255 + 1563.3 \times 0.2873 = 1{,}374.01 \textit{ Ans.}$$

With the same conditions in Fig. 167, suppose the problem is to find what force applied to the wedge will cause impending downward motion of the weight. First, set the wedge and weight out as free bodies (Fig. 167). Since the motion of the weight is downward, the resultant reaction of the surfaces of the wedges must act on the side of the normal opposite to that acted on in the preceding problem. Also, since the wedge tends to move to the left, the reactions must act on the left side of the normals. The force triangles are now drawn. The one for the wedge shows F acting to the left. The mechanism is self-locking, and it requires a force to pull the wedge from under the weight.

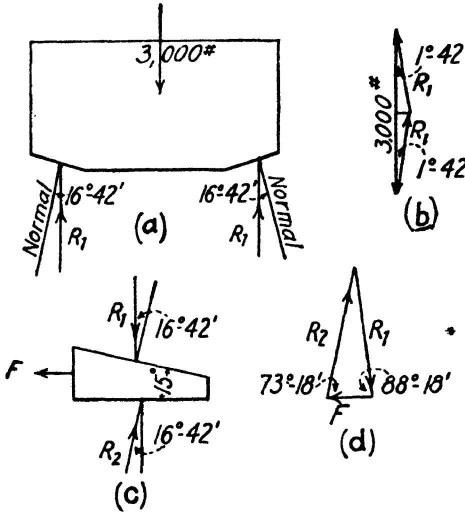


FIG. 167.

Example.—Suppose the angle of the wedge in the previous problem is 25° and the angle of friction of the rubbing surfaces is 10°. Find the force F to allow the weight to be lowered at a uniform speed, or what amounts to the same thing, to have downward motion impending (Fig. 168). As in the preceding problems, free-body drawings are made for the weight and the wedge. The force triangles are then constructed. The force diagram for the wedge shows that F acts to the right; *i.e.*, the mechanism is not self-locking. A force must be exerted on the wedge to prevent downward motion of the weight. The solution of the triangles is left for the student.

87. Journal Friction.—When a shaft or axle rotates in its bearings, a suitable amount of lubrication is usually applied in order to reduce the effect of friction. However, since there is sliding between surfaces in contact, a force of friction is present and must be recognized. Owing to the lubrication the value of *f* is reduced to 0.1 or less of the value of *f* for a similar surface unlubricated. Thus, if *f* is 0.15 for certain dry surfaces, this value would reduce to 0.015 or less for the same surface if lubricated. The effect of the friction is to cause wear on the bearings as well as to increase the turning moment needed to continue rotation. Figure 169 shows a shaft and bearing. The resultant of the friction and the normal reaction will be equal

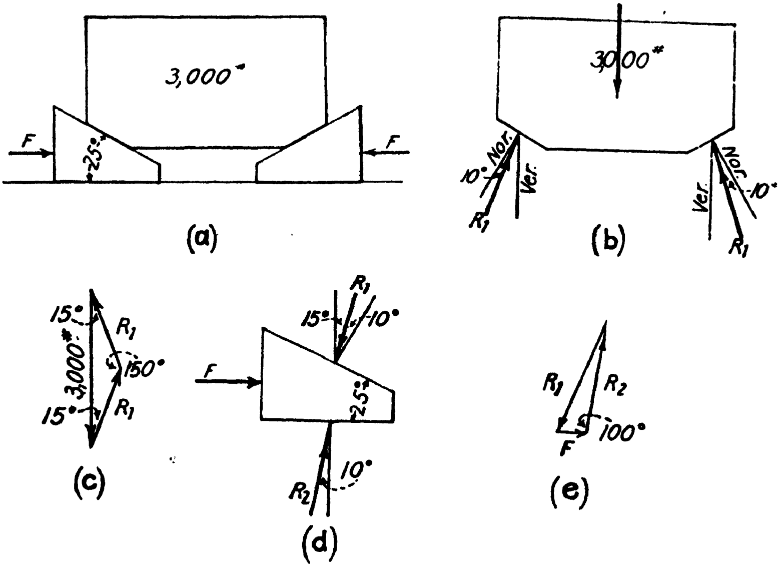


FIG. 168.

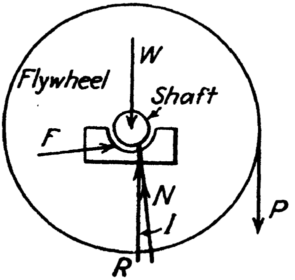


FIG. 169.

and opposite to the resultant of the weight and the pull. The friction will therefore be at a slight angle with the horizontal in order to satisfy the requirements for equilibrium. In other words, the shaft will at first climb up on the bearing until a point is reached where the forces are in equilibrium.

Example 1.—A flywheel weighing 1 ton is 6 ft. in diameter. The coefficient of friction is 0.01. What tangential force applied to the rim will cause rotation? The diameter of the shaft on which the flywheel is mounted is 6 in.

Solution.

$$W = 2{,}000, \qquad F = 0.01 \times 2{,}000 = 20$$

Taking moments about the axis of the shaft,

$$P \times 36 = 20 \times 3$$
$$P = \frac{20 \times 3}{36} = 1.67 \text{ lb. } \textit{Ans.}$$

This is the force that may be assumed to be sufficient to turn the wheel. There is a slight omission in that the actual normal force is equal to 2,000 + P. But, in this case, it can be seen that there is no appreciable error.

Example 2.—A pulley 27 in. in diameter weighs 80 lb. It is mounted on a 2-in. axle. A weight of 1,000 lb. is to be lifted. If $f = 0.1$, what pull will be needed? The value of f implies poor lubrication.

Solution.

$$N = 80 + 1{,}000 + P$$
$$F = 0.1(80 + 1{,}000 + P)$$

Taking moments about the center of the axle,

$$0.1(80 + 1{,}000 + P) \times 1 = P \times 13.5 - 1{,}000 \times 13.5$$
$$80 + 1{,}000 + P = (P - 1{,}000)135 = 135P - 135{,}000$$
$$135P - P = 135{,}000 + 1{,}080 = 136{,}080$$
$$134P = 136{,}080$$
$$P = \frac{136{,}080}{134} = 1{,}015.52 \text{ lb. } \textit{Ans.}$$

88. Rolling Resistance.—A cylindrical body such as a wheel rolling on a plane surface would be in contact along a line corresponding to the element of the cylinder, if no deformation took place. Actually, since all materials are elastic, the wheel flattens slightly and at the same time causes a slight depression in the surface on which it rests, with the net result that there is a slight area of contact. Figure 170 illustrates the situation if no deformation took place, and Fig. 170*a* shows the actual situation but with the deformation magnified. The net effect of the deformation is to introduce a resistance which may be compared with that of a wheel being moved over an obstacle as in Example 1, Art. 73.

The distance *a* in Fig. 170*a* is called the *coefficient of rolling resistance.* These distances have been determined for various materials and vary from 0.03 for wood to 0.007 for steel. There are intermediate values depending on the exact composition of the materials as well as on the particular materials that are in contact at the time. Rolling resistance is the principal force that must be overcome whenever a railroad car or a streetcar is moved along a level track or when a truck or an automobile is moved along a level pavement.

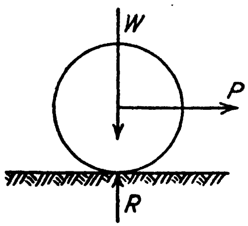


Fig. 170.

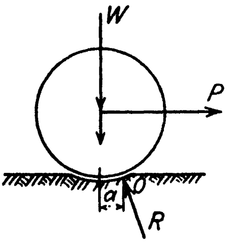


Fig. 170*a*.

The force required to cause the rotation of the wheel can be found readily by taking moments about point *O*. Then

$$P \times r \cos \Theta = W \times a$$

But Θ is small.

$$\therefore P \times r = W \times a$$

$$P = \frac{Wa}{r} \tag{28}$$

where a and r are in inches.

Example.—A steel wheel 36 in. in diameter and weighing 700 lb. is moved on a steel track by means of a pull applied through the center of the wheel and parallel to the track. If $a = 0.01$, what is the amount of the pull?

Solution.—$W = 700$, $a = 0.01$. Using Eq. (28), we have

$$P = \frac{700 \times 0.01}{18} = 0.389 \text{ lb. } \textit{Ans.}$$

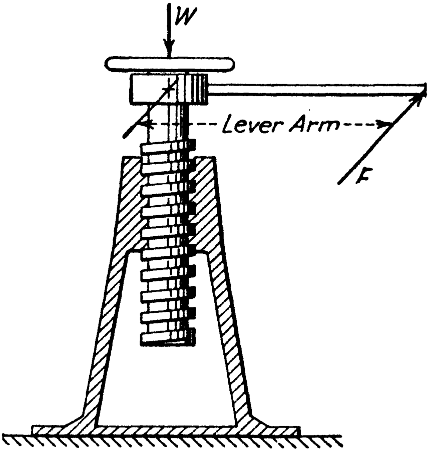


Fig. 171.

89. Jackscrew.—In this machine, the core, which is a threaded screw, is made to turn in a stationary frame or base (Fig. 171). As the force is applied to the lever, the screw turns and moves upward, lifting the load. The thread in the frame is an inclined plane. The weight supported by the screw is transmitted to the inclined plane. As shown in Fig. 171*a*, if a single thread is set out as a free body, the representation of forces will be typical.

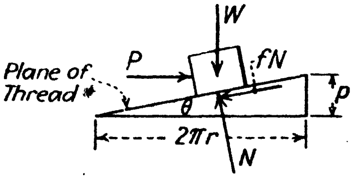


Fig. 171*a*.

From $\Sigma X = 0$,

$$P = N \sin \theta + fN \cos \theta$$

From $\Sigma Y = 0$,

$$N \cos \Theta - fN \sin \Theta = W$$

$$\frac{P}{W} = \frac{\sin \Theta + f \cos \Theta}{\cos \Theta - f \sin \Theta} = \tan (\Theta + I)$$

To produce motion, the screw must be rotated by the force F. Then

$$Fl = Pr = W \tan (\Theta + I) \times r$$

Example.—A jackscrew with a mean radius of ¾ in. has a pitch of ¼ in. With $f = 0.1$, what force applied to the end of a 20-in. lever will lift 2 tons?

Solution.

$$\tan \Theta = \frac{0.25}{2\pi 0.75} = 0.053, \qquad \Theta = 3°2''$$

$$\tan I = 0.1, \qquad I = 5°43''$$

$$F \times 20 = 2 \times 2{,}000 \tan (8°45'') \times 0.75$$

$$F = \frac{2 \times 2{,}000 \times 0.1539 \times 0.75}{20} = 23.08 \text{ lb. } \textit{Ans.}$$

Problems

151. The weight of a wooden block resting on a horizontal wooden floor is 60 lb. What horizontal pull is needed to start the block from rest if the coefficient of friction is 0.3?

152. A 50-lb. body rests on a plane making an angle of 30° with the horizontal. What force parallel to the plane is needed to start the body up the plane? The coefficient of friction is 0.3.

153. In the preceding problem, what force parallel to the plane will keep the body from sliding down the plane? What horizontal force is needed to start the body up the plane?

154. A body weighing 100 lb. rests on a horizontal plane. A horizontal force of 20 lb. is just enough to cause motion to impend. What is the coefficient of friction?

155. A body weighing 100 lb. rests on a plane making an angle of 10° with the horizontal. A horizontal force of 30 lb. is just sufficient to cause motion to impend up the plane. What is the coefficient of friction?

156. A body weighing 100 lb. rests on an inclined plane. A horizontal force of 30 lb. is just sufficient to cause motion to impend up the plane. If $f = 0.1$, what angle does the plane make with the horizontal?

157. Will the bodies in Fig. 172 slide? What is the tension in the cord connecting the two bodies? What force is needed to start the 200-lb. body to the right? The coefficient of friction is 0.25.

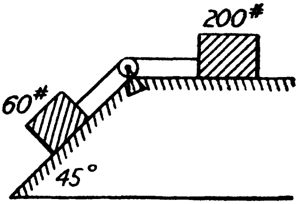


FIG. 172.

158. In the friction drive shown in Fig. 173, the wheel A transmits a turning motion of 200 in.-lb. to the wheel B. If the coefficient between the surfaces of the wheels in contact is 0.35, what must be the pressure between the wheels, and what are the reactions at the bearings C and D?

159. A car weighing 2,500 lb. is 56 in. wide, center to center of wheels, and 110 in. long, center to center of axles. The center of gravity is 30 in. above the ground and 65 in. in front of the rear axle. If the coefficient of friction for the brake is 0.3 and the brakes are set so there can be no skidding, what is the steepest grade that the car can descend with uniform velocity? Solve problem using four-wheel brakes, also two-wheel brakes. Assume f between wheel and roadway to be 0.35.

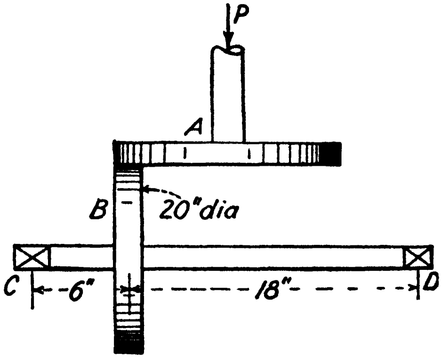


FIG. 173.

160. A drag is attached to a tractor by means of a chain. The point of attachment of the chain to the drag is 9 in. above the ground and to the tractor 30 in. above the ground. If the coefficient of friction between the drag and the ground is 0.75, what length chain will require the least pull upon it?

161. A weight of 1,000 lb. is to be raised by means of a wedge which is moved by a horizontal force P (Fig. 174). The weight is compelled to move vertically, owing to the horizontal force F. What are the amounts of the horizontal forces P and F if $f = 0.25$ for all surfaces? What would be the force P if weight is lowered?

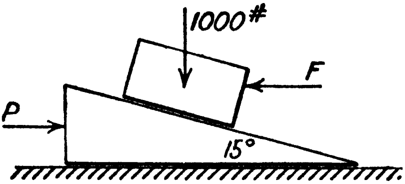


Fig. 174.

162. What load Q can be supported by the force of 600 lb. if f for all surfaces is 0.4 (Fig. 175)?

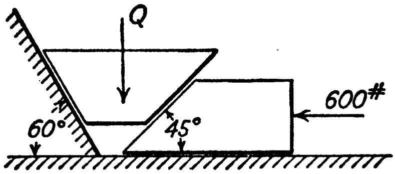


Fig. 175.

163. What will be the amounts of the forces P needed to lift a body weighing 200,000 lb. (Fig. 176)? The coefficient of friction is 0.4.

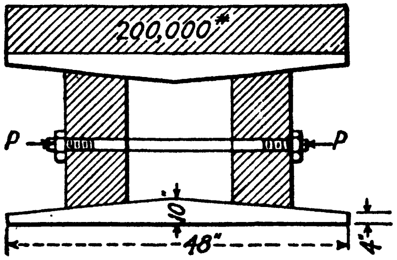


Fig. 176.

164. If the ultimate tensile strength of the wooden piece A shown in Fig. 177 is 3,200 lb. per sq. in., what force must be applied to the wedge to rupture the piece A? The coefficient of friction between the wedge and the piece is 0.3.

165. A body weighing 100 lb. rests on a horizontal floor. If $f = 0.4$, what upward pull at an angle of 30° with the horizontal is needed to start the body?

166. A body weighing 200 lb. rests on a plane making an angle of 20° with the horizontal angle. What must be the value of f to keep the body from sliding down the plane?

167. What least pull is needed to start a body weighing 40 lb. up a plane making an angle of 25° with the horizontal, if $f = 0.3$?

168. A body weighing 200 lb. is placed on a plane making an angle of 20° with the horizontal. If $f = 0.4$, will the body slide?

169. A ladder 10 ft. long and weighing 40 lb. is placed with the lower end resting on a rough floor and the upper end against a wall at a point 8 ft. above the floor. If the value of $f = 0.2$, will the ladder remain in position? The weight of the ladder is assumed to act at the mid-point of its length.

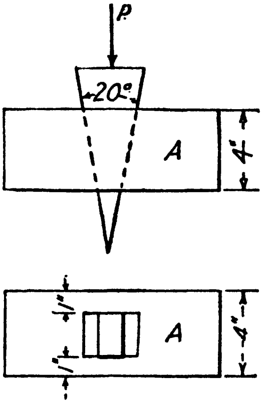


FIG. 177.

170. A freight car weighing 80 tons is left standing on a siding with a 4 per cent grade. What must be the normal pressure of the brake shoes against each of the eight 33-in. wheels to keep the car from coasting down the grade? The coefficient of friction between the wheels and the rails is 0.3 and that between the wheels and the brake shoes is 0.5.

171. What is the amount of the drawbar pull needed to move a 50-ton freight car with wheels 33 in. in diameter, if the coefficient of rolling resistance is 0.3?

172. The mean radius of the screw of a square-threaded jackscrew is 1 in. The pitch of the threads is 0.3 in. If the coefficient of friction is 0.12, what force applied to the end of a lever 24 in. long is needed to raise a weight of 4,000 lb.?

CHAPTER VII

CONCURRENT—NON-COPLANAR FORCES

90. Resultant Equilibrium.—When forces are not in the same plane, they are called *non-coplanar.* To find the resultant of such forces, it is best to resolve each force into components along three axes making angles of 90° with each other.

Let Ox, Oy, and Oz be the axes and O the point through which the forces pass. Take OE as one of the forces, making the angles α, β, and γ with the axes (Fig. 178). Then the components or projections of $OE = P$ give $P_x = P \cos \alpha$, $P_y = P \cos \beta$, and $P_z = P \cos \gamma$. But the three projections OG, OA, and OC are the three edges of a rectangular parallelepiped of which the force P is the diagonal. From geometry, the square on the diagonal is equal to the sum of the squares of the three edges. Then,

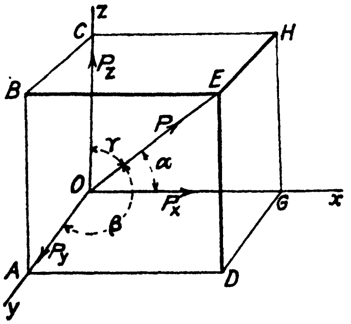


Fig. 178.

$$P^2 = P_x^2 + P_y^2 + P_z^2 = P^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$$

But this cannot be true unless

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Angles α, β, and γ are called the "direction angles of the force P," and the foregoing equation gives the condition that their cosines must satisfy. For example, if $\alpha = 60°$, and $\beta = 45°$,

$$
\begin{aligned}
\cos^2 60^\circ + \cos^2 45^\circ + \cos^2 \gamma &= 1 \\
0.25 + 0.5 + \cos^2 \gamma &= 1 \\
\cos^2 \gamma &= 0.25 \\
\cos\ \gamma &= 0.5 \\
\gamma &= 60^\circ
\end{aligned}
$$

If each of the concurrent system of forces is resolved into their x, y, and z components, the original system is replaced by three sets of forces acting along the x, y, and z axes, respectively. Since all the forces along the x axis are in the same straight line, they may be added algebraically, and their sum represented by ΣP_x. Similarly, those along the y axis and the z axis may be added algebraically and their sums represented by ΣP_y, and ΣP_z, respectively. Their resultant, the diagonal of the parallelepiped formed by ΣP_x, ΣP_y, and ΣP_z, is equal to

$$R = \sqrt{(\Sigma P_x)^2 + (\Sigma P_y)^2 + (\Sigma P_z)^2} \qquad (29)$$

Example 1.—What will be the resultant if the x, y, and z components of Fig. 178 are, respectively, 10, 20, and 15 lb.? Assume the values of the angles as given in the text.

Solution.—The resultant of the three forces may be represented as the diagonal of a parallelepiped whose three sides have, respectively, the values of the three forces given. Then, in accordance with the explanation,

$$\text{Resultant} = R = \sqrt{10^2 + 20^2 + 15^2} = 26.9 \text{ lb. } \textit{Ans.}$$

Example 2.—The force directed along the line OE in Fig. 178 is 100 lb. What will be the components along the three axes.

Solution.

$$\alpha = 60^\circ, \qquad \beta = 45^\circ, \qquad \gamma = 60^\circ$$

$$
\begin{aligned}
P_x &= 100 \cos \alpha = 100 \cos 60^\circ = 100 \times 0.5 = 50 \text{ lb.} \\
P_y &= 100 \cos \beta = 100 \cos 45^\circ = 100 \times 0.707 = 70.7 \text{ lb.} \\
P_z &= 100 \cos \gamma = 100 \cos 60^\circ = 100 \times 0.5 = 50 \text{ lb. } \textit{Ans.}
\end{aligned}
$$

91. Condition for Equilibrium.—*When forces are in equilibrium, their resultant is zero.* That is,

$$\sqrt{(\Sigma P_x)^2 + (\Sigma P_y)^2 + (\Sigma P_z)^2} = 0$$

But each quantity under the radical sign is positive, being a square. The sum. of positive numbers cannot be zero unless each one is equal to zero. Then,

$$\Sigma P_x = 0, \qquad \Sigma P_y = 0, \qquad \Sigma P_z = 0 \tag{30}$$

This result gives three equations of condition, from which three unknowns may be determined.

If the force polygon were drawn for a set of concurrent non-coplanar forces that were in equilibrium, the polygon would close. Then the projection of this polygon on any plane would be a closed plane polygon. But since it is a closed figure, the forces that form it are in equilibrium. We then have coplanar forces in equilibrium, and the principles of Chap. IV may be applied.

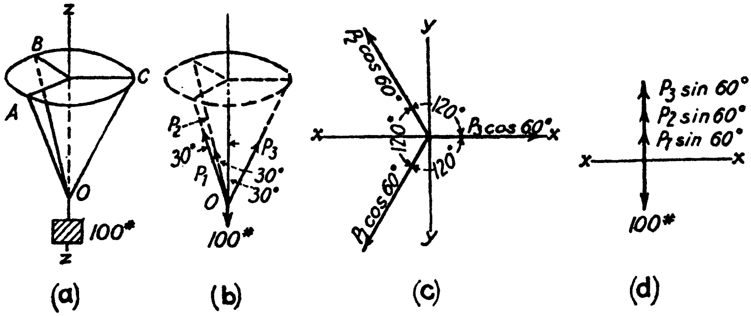


FIG. 179.

$\Sigma P_x = 0$ and $\Sigma P_y = 0$ are the equations of condition that must be satisfied when concurrent forces in equilibrium are projected on the x and y axes and also on the xy plane. Also, ΣP_z is the equation that the projection of such forces on the z axis must satisfy.

The forces might also be projected on the xz plane and the y axis or the yz plane and the x axis. The conclusion may be summarized as follows: *When a set of concurrent non-coplanar forces are in equilibrium, project all the forces on a convenient plane and also on an axis perpendicular to the plane. Then, form $\Sigma P_x = 0$, $\Sigma P_y = 0$, $\Sigma P_z = 0$, and solve simultaneously.*

Example 1.—A 100-lb. body is suspended from the ceiling by three cords of equal length. The cords make angles of 30° with the vertical, and they are attached to the ceiling at points equally spaced on the arc of a circle. Find the tensions in the cords (Fig. 179).

Solution.—Let the tensions in *OA*, *OB*, and *OC* be P_1, P_2, and P_3. Set point *O* out as a free body (Fig. 179*b*). Now, project all the forces on a horizontal plane and a vertical axis (Figs. 179*c* and 179*d*).

$$\Sigma P_x = -P_1 \cos 60° \cos 60° - P_2 \cos 60° \cos 60° + P_3 \cos 60° = 0 \quad (30_1)$$
$$\Sigma P_y = -P_1 \cos 60° \sin 60° + P_2 \cos 60° \sin 60° = 0 \quad (30_2)$$
$$\Sigma P_z = (P_1 + P_2 + P_3) \sin 60° - 100 = 0 \quad (30_3)$$

From Eq. (30_2) above,

$$P_1 = P_2$$

and from Eq. (30_1) above,

$$P_1 = P_2 = P_3$$

From Eq. (30_3) above,

$$3P \times 0.866 = 100$$
$$P_1 = P_2 = P_3 = \frac{100}{2.6} = 38.3 \text{ lb. } \textit{Ans.}$$

Example 2.—A 200-lb. weight is suspended from the wall by means of a bracket, as shown in Fig. 180*a*. Find the stresses in *OA*, *OB*, and *OD*.

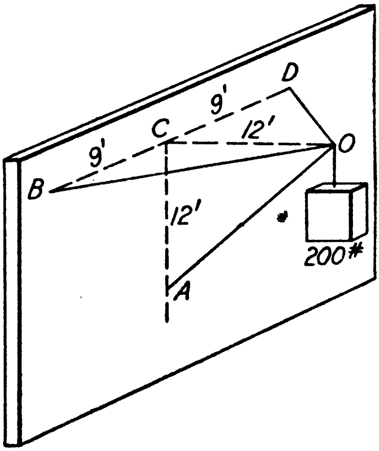


Fig. 180*a*.

Solution.—Let the stresses be represented by P_1, P_2, and P_3. Take point *O* as a free body. First project the forces onto a vertical axis through O. Since P_2 and P_3 are perpendicular to this axis, their projection is zero. Then

$$P_1 \sin \angle COA - 200 = 0$$

But

$$\angle COA = 45°$$

$$\therefore P_1 = \frac{200}{0.707} = 282.7 \text{ lb.}$$

Now project the forces on a horizontal plane through O (Fig. 180*b*). Since P_2 and P_3 make equal angles with P_1, they are equal to each other.

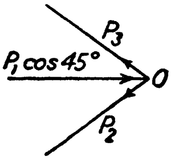


FIG. 180*b*.

$$\Sigma P_x = -2P_2 \cos \angle COD + 282.7 \cos 45° = 0$$

$$DO = \sqrt{9^2 + 12^2} = 15$$

$$\therefore 2 \times \tfrac{4}{5}P_2 = 200$$

$$P_2 = P_3 = 125 \text{ lb.}$$

Example 3.—A 500-lb. weight is suspended from the wall by means of a bracket, as shown in Fig. 181. Find the stresses in the members OA, OB, and OD.

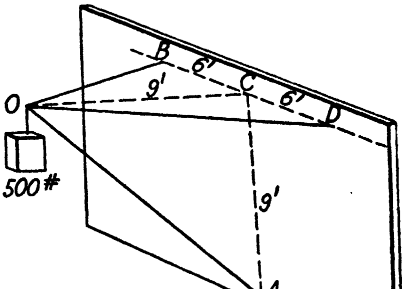


FIG. 181.

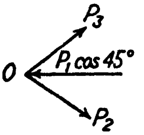


FIG. 181*a*.

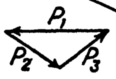


FIG. 181*b*.

Solution.—Represent the stresses by P_1, P_2, P_3. Take BOD as a free body. The forces acting on BOD are P_1, 500, and the reactions at B and D. Taking moments about line BD, we have

$$P_1 \times 9 \sin \angle COA - 500 \times 9 = 0$$

$$P_1 = \frac{500 \times 9}{9 \sin 45°} = \frac{500}{0.707} = 707 \text{ lb. } \textit{Ans.}$$

Now, project the three stresses on a horizontal plane, with the point *O* as the free body (Fig. 181*a*). The force triangle will appear as shown in

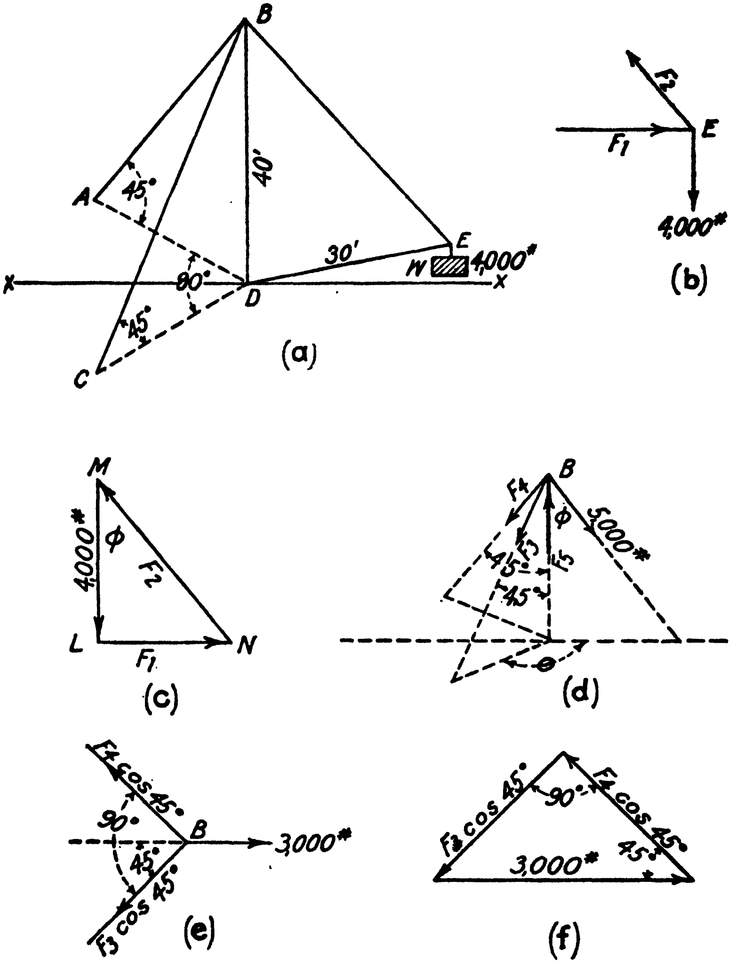


Fig. 182.

Fig. 181*b*. Since this is an isosceles triangle, it may be solved by dividing it into two right triangles. Then

$$P_3 \cos 45° + P_2 \cos 45° = P_1 \times \cos 45°$$

But $P_3 = P_2$.

$$2P_3 \times 0.707 = 707 \times 0.707$$

$$P_3 = \frac{707}{2} = 353.5 \text{ lb. } \textit{Ans.}$$

Example 4.—The boom of the derrick shown in Fig. 182 can swing from the horizontal to 25° from the vertical and through the larger angle *CDA*. With the boom horizontal and bisecting the angle *CDA*, find the stresses in all members of the derrick.

Solution.—Since all members are two-force pieces, their lines of action are known. At point *B*, there are four forces in equilibrium. Since they are non-coplanar, there are but three equations of condition that can be formed, and the solution cannot start at this point.

At point *E*, there are three coplanar forces, one of which is known. Since for this case two equations exist, the solution is possible. Set point *E* out as a free body, and draw the force triangle *LMN*. Triangle *LMN* is similar to triangle *DBE*, and the stresses in *BE* and *DE* are proportional to their lengths. By proportion,

$$F_1 : 4{,}000 :: DE : DB :: 30 : 40$$
$$F_1 = 4{,}000 \times \tfrac{30}{40} = 3{,}000 \text{ lb. compression}$$
$$F_2 : 4{,}000 :: BE : BD :: 50 : 40$$
$$F_2 = 4{,}000 \times \tfrac{50}{40} = 5{,}000 \text{ lb. tension}$$

Point *B* may now be set out as a free body, since the stress in *BE* is known and there are but three unknowns left (Fig. 182*d*). First project the forces on a horizontal plane through *B*, as shown in Fig. 182*e*. The projection of F_5 is zero since it is perpendicular to the plane of projection. Now, draw the force triangle (Fig. 182*f*). Since it is an isosceles triangle, F_3 equals F_4.

$$3{,}000 \cos 45° = F_3 \cos 45°$$
$$F_3 = 3{,}000 \text{ lb.} = F_4$$

Now, project all forces at point *B* onto a vertical line through *B*.

$$F_3 \cos 45° + F_4 \cos 45° + 5{,}000 \cos \varphi = F_5$$

But

$$F_3 = F_4, \qquad \text{and } \cos \varphi = \tfrac{4}{5}$$

Then

$$2 \times 3{,}000 \cos 45° + 4{,}000 = F_5$$
$$F_5 = 8{,}282 \text{ lb. compression}$$

When a derrick is to be designed, the maximum stresses that exist in every member are to be found for a given load while the boom moves through its entire range. Since the force triangle *LMN* is similar to *BDE*, two of the sides of which are of fixed lengths, the stress in *BE* will be a maximum when *BDE* is a right angle, that is, when the boom is horizontal.

Now the maximum stresses will occur in *AB*, *CB*, and *BD* when the stress in *BE* is a maximum. It now remains to find, when the boom *DE* is horizontal, what angle it must make with *CD* to give maximum stresses in the mast and legs. Let $\angle CDE = \theta$. Set point *B* out as a free body. Now project, all the forces onto a hori-

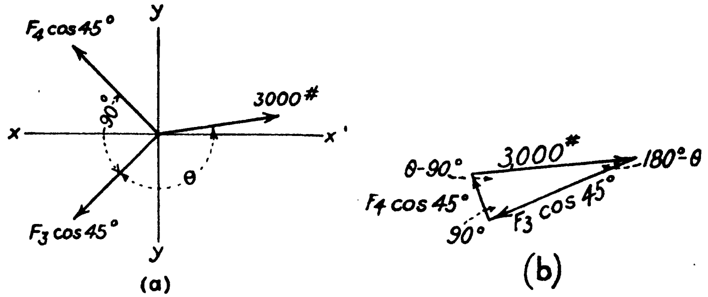


FIG. 183.

zontal plane (Fig. 183*a*). Next, draw the force triangle (Fig. 183*b*). By definition,

$$\sin(\theta - 90) = \frac{F_3 \cos 45°}{3{,}000}$$

$$F_3 = \frac{3{,}000 \sin(\theta - 90°)}{\cos 45°}$$

Now F_3 is greatest when $\sin(\theta - 90)$ is greatest, *viz.*,

$$\sin(\theta - 90) = 1$$
$$\theta - 90 = 90$$
$$\theta = 180°$$

This makes the plane of the boom perpendicular to the plane of the leg *AB*. Although this case is special in that the planes of the legs are perpendicular to each other, the same method of procedure for any angle *CDA* will lead to the same conclusion.

The maximum tension is found in one leg of a derrick when the plane of the boom is perpendicular to the plane of the other leg.

Substituting $\theta = 180°$ in the equation for F_3, we have

$$F_3 = \frac{3{,}000 \sin 90°}{0.707} = 4{,}242 \text{ lb.}$$

$$F_4 = \frac{3{,}000 \sin 0°}{0.707} = 0$$

The theorem for maximum tension could be applied as well to maximum compression, if it were possible to swing the plane of the boom into the smaller angle CDA in case it was greater than 90°. However, the maximum compression will be developed in either leg when the plane of the boom coincides with the plane of that leg *and is on the same side of the mast.* Again, set point B out as a free body with the boom in the position stated, and project the forces onto a horizontal plane (Fig. 184). By inspection,

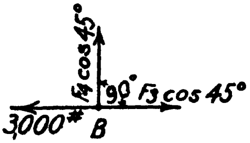


FIG. 184.

$$F_4 = 0$$

and

$$F_3 \cos 45^\circ = 3{,}000 \text{ lb.}$$
$$F_3 = 4{,}242 \text{ lb.}$$

To obtain the stress in DB for each of the foregoing cases, project B onto the vertical axis DB. Then

$$5{,}000 \cos \varphi + 4{,}242 \cos 45^\circ = F_5$$
$$4{,}000 + 3{,}000 = F_5$$
$$F_5 = 7{,}000$$

This result is less than the value obtained when the plane of the boom bisected angle CDA. To prove this, let $\angle CDE = \Theta$ (Fig. 183*d*). Then

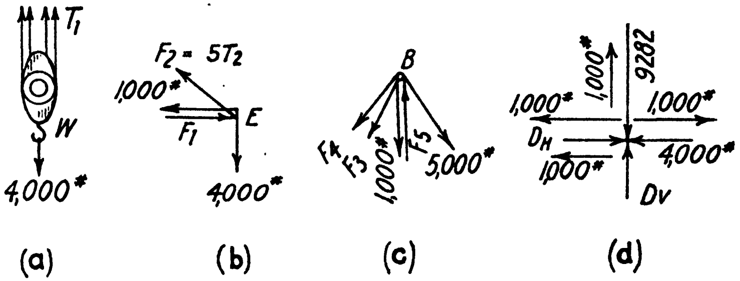
$$F_3 = \frac{3{,}000 \sin (\Theta - 90)}{\cos 45^\circ} = -\frac{3{,}000 \cos \Theta}{0.707}$$
$$F_4 = \frac{3{,}000 \sin (180 - \Theta)}{\cos 45^\circ} = \frac{3{,}000 \sin \Theta}{0.707}$$

From the vertical projection,

$$F_5 = 4{,}000 + F_3 \cos 45 + F_4 \cos 45$$
$$= 4{,}000 + 3{,}000 (-\cos \Theta + \sin \Theta)$$

Now F_5 is a maximum when $\sin \Theta - \cos \Theta$ is greatest. This condition occurs when both functions are numerically equal and $\cos \Theta$ is negative, *i.e.*, when $\Theta = 135^\circ$. When angle CDA is not 90°, the same line of reasoning will show maximum stress in the mast when the plane of the boom bisects the angle between the legs.

Example 5.—In Example 4, in order to lift the weight vertically with the boom stationary, it is connected to E by a block and tackle. Four lengths of cable run from E to W. One end is fastened at E, and the other runs along DE through a pulley at D and then to a winding drum. Between B and E, there are five strands of another cable, the free end of which runs down BD to a pulley at D and then to a second winding drum. Find maximum stresses in all members, and the ground reaction at D.



(a) (b) (c) (d)

FIG. 185.

Solution.—Set W out as a free body (Fig. 185a). If we neglect friction on the pulleys, the tension in the cable is the same at all points and each of the upward forces is T_1. By equilibrium,

$$4T_1 = 4{,}000$$
$$T_1 = 1{,}000 \text{ lb.}$$

Now, set point E out as a free body (Fig. 185b). By taking $\Sigma Y = 0$, $F_2 = 5{,}000$ as in Example 4. But this is carried by five cables, and

$$5T_2 = 5{,}000$$
$$T_2 = 1{,}000$$

From $\Sigma X = 0$,

$$F_1 = 1{,}000 + 3{,}000 = 4{,}000 \text{ lb.}$$

Now, set point B out as a free body (Fig. 185c). Projecting the forces on a horizontal plane, we see that F_3 and F_4 have the same values as in Example 4. Projection on a vertical axis shows $F_5 =$ 9,282 lb., or 1,000 lb. greater than in the preceding example.

For determining ground reactions, set *D* out as a free body (Fig. 185*d*). The ground reaction not being known, the components are used.

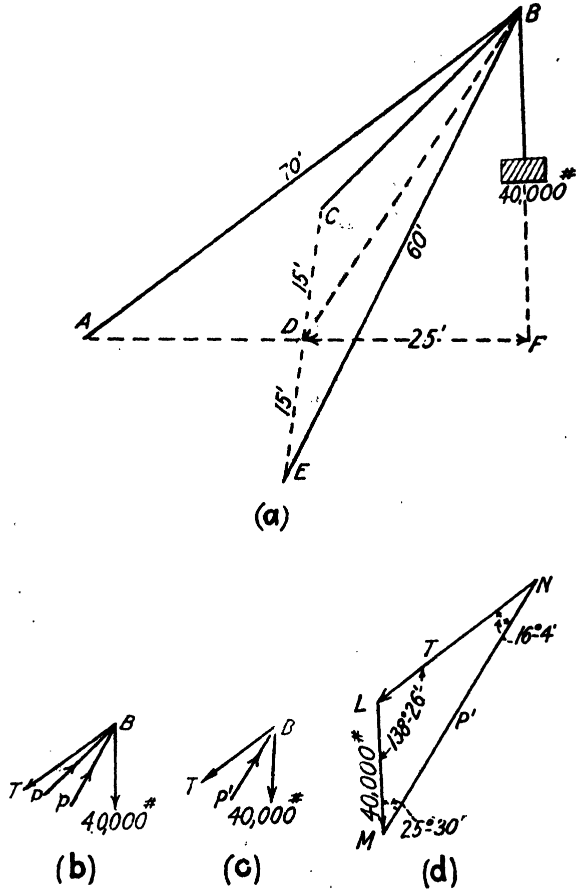


Fig. 186.

By $\Sigma Y = 0$ and $\Sigma X = 0$,

$$D_v = 9{,}282 - 1{,}000 = 8{,}282$$

$$D_h = -1{,}000 + 1{,}000 + 1{,}000 + 4{,}000 = 5{,}000$$

$$D = \sqrt{(8{,}282)^2 + (5{,}000)^2} = 9{,}673 \text{ lb.}$$

$$\theta = \tan^{-1}\frac{(8{,}282)}{(5{,}000)} = \tan^{-1}(1.6564)$$

$$\theta = 58°53' \text{ with the horizontal.}$$

Example 6.—Figures 186a to d show a shear-leg crane lifting a 40,000-lb. load. The legs are 60 ft. long, and 30 ft. apart at their lower ends. The backstay is 70 ft. long. All members are pin-connected, and A, E, and C are in the same plane. Find the stresses in the members.

Solution.—All members are two-force pieces and concurrent at B. Set B out as a free body. Project all forces onto a vertical plane ABF. Since the two forces P are in a plane perpendicular to the plane of projection, they project into P' their resultant.

Next, form the force triangle LMN. Angles M, L, and N must now be found.

$$\overline{BD}^2 = \overline{60}^2 - \overline{15}^2 = 3{,}375$$
$$BD = 58.1$$
$$\overline{BF}^2 = \overline{BD}^2 - \overline{25}^2 = 2{,}750$$
$$BF = 52.4 \text{ ft.}$$
$$\cos \angle ABF = \frac{52.4}{70} = 0.7486$$
$$\angle ABF = 41°34'$$
$$L = 180 - 41°34' = 138°26'$$
$$\tan \angle DBF = \tan M = \frac{25}{52.4} = 0.4771$$
$$M = 25°30'$$
$$N = 180 - (138°26' + 25°30') = 16°4'$$

From the triangle LMN,

$$\frac{T}{40{,}000} = \frac{\sin 25°30'}{\sin 16°4'}$$
$$T = 62{,}460 \text{ lb.}$$

Also,

$$\frac{P'}{40{,}000} = \frac{\sin 138°26'}{\sin 16°4'}$$
$$P' = 95{,}920 \text{ lb.}$$

Since P' is the resultant of the thrusts P, it will equal the sum of the components along BD, or

$$2P \cos \angle DBC = 95{,}920$$
$$P = \frac{95{,}920}{2 \times (58.1/60)} = 49{,}530 \text{ lb.}$$

Example 7.—A tripod has legs 12 ft. long. The lower ends of the legs are placed at the corners of an equilateral triangle 6 ft. on a side. What are the stresses in the legs when a load of 1,000 lb. is placed on top of the tripod?

Solution.—Figure 187 shows a top view of legs and base. The point D' is the projection of the vertex of the tripod on the plane of the base. Since the sides of the base are equal in length and the legs are equal in length, this projection falls at the centroid, or median point of the triangle. The angle that each leg makes with the vertical can be determined from the triangle ADD'.

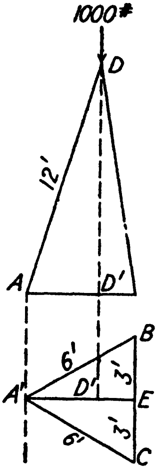


FIG. 187.

$$AD' \doteq \tfrac{2}{3} \times AE = \tfrac{2}{3} \times (6^2 - 3^2)$$
$$AD' = \tfrac{2}{3} \times 5.19 = 3.46 \text{ ft.}$$
$$DD' = (\overline{AD}^2 - \overline{AD'}^2) = 11.49 \text{ ft.}$$

Since the legs are symmetrically arranged and make equal angles with the vertical, then, from $\Sigma Y = 0$,

$$3AD \cos \Theta = 1{,}000$$
$$\frac{3AD11.49}{12} = 1{,}000$$
$$AD = \frac{12 \times 1{,}000}{3 \times 11.49} = \frac{4{,}000}{11.49} = 348.2 \text{ lb. } Ans.$$

Problems

173. A load of 10 tons is to be lifted by means of a contractor's derrick, as shown in Fig. 189. If the vertical plane containing the boom bisects the angle ADB, what reactions must be provided at points A and B to keep the derrick from tipping? The boom weighs 4,000 lb. and makes an angle of 45° with the horizontal. When the boom swings into the plane BCD, what will be the stress in the leg BC?

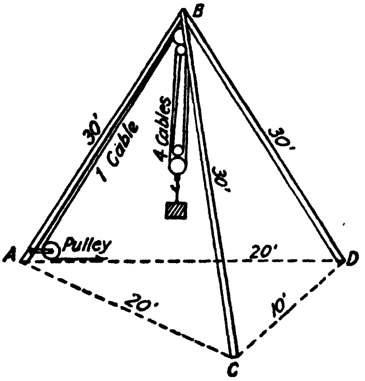


FIG. 188.

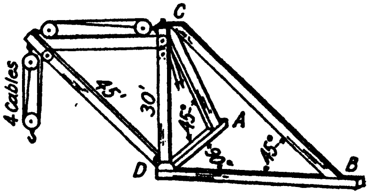


FIG. 189.

174. Figure 188 shows a tripod used in unloading heavy machinery. If a compressive stress of 200 lb. per sq. in. is allowed, what should be the size of square members if the load is 4 tons?

175. What are the stresses in all parts of the shear-leg crane shown in Fig. 192 if a naval gun weighing 50 tons is lifted?

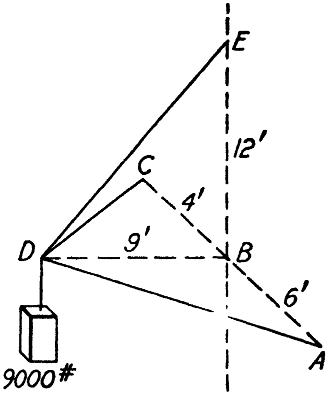


Fig. 190.

176. Design all members in the bracket shown in Fig. 190. *DE* is intended to be a steel member with an allowable stress of 5,000 lb. per sq. in., and *AD* and *CD* are wooden members with stresses not exceeding 1,000 lb. per sq. in.

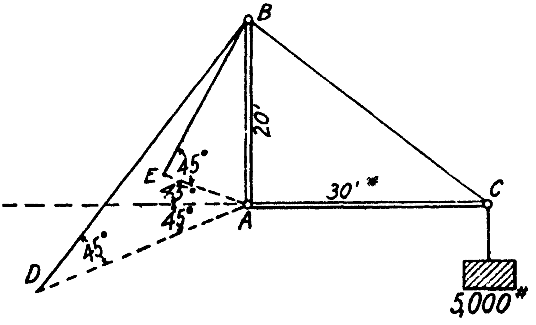


Fig. 191.

177. What size of steel rod is needed for member *BD* in the derrick in Fig. 191, with boom *AC* in the position shown? What would be the stress in *BD* if boom *AC* were rotated into plane *ABD*?

178. Figure 193 shows a derrick commonly used in a quarry. The mast and boom are pivoted at the top and the bottom so that they can be rotated through 360°. The angle that the boom makes with the vertical is also adjustable. If a load of rock weighing 10 tons is raised, what is the stress in the boom when it makes an angle of 45° with the vertical? The effective length of the mast and boom may be taken as 24 and 18 ft., respectively.

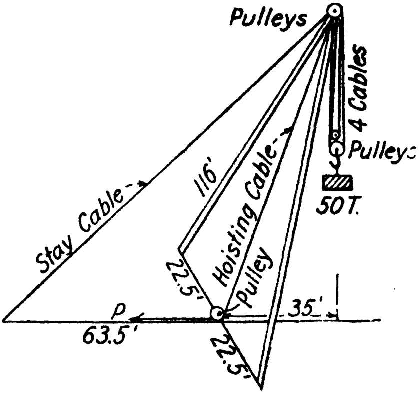


Fig. 192.

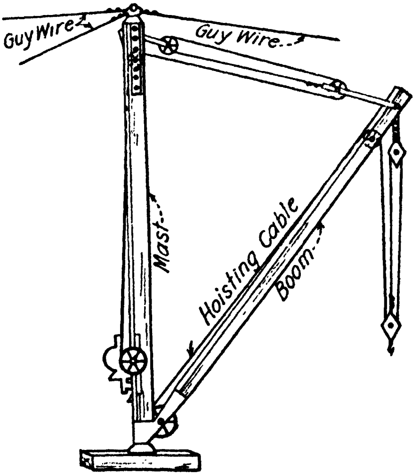
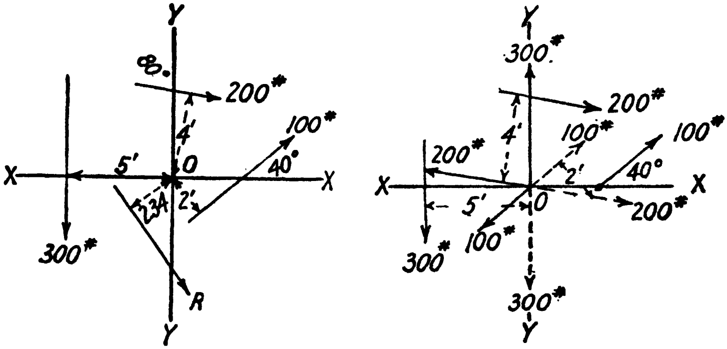


Fig. 193.

CHAPTER VIII

NONCONCURRENT COPLANAR FORCES

92. Discussion.—In Chap. IV, on concurrent forces, it was shown that the resultant of a set of concurrent forces was a single force determined by ΣX and ΣY. Also, in Chap. V, on parallel forces, the resultant was found to be a single force and a couple, determined by ΣX and ΣY and ΣM. In this chapter, it will be shown that the resultant of nonconcurrent nonparallel forces is also a single force and a couple. The principle will be illustrated by an example before taking up the general proof. Figure 194*a* shows three forces in action. Find the resultant in magnitude and position with reference to point *O*. At *O* (Fig. 194*b*), introduce two forces of 100 lb. each, opposite in direction, and parallel to the original 100-lb. force. This does not change conditions, for the forces introduced are in equilibrium. The two 100-lb. forces drawn in full line form a couple with a moment arm of 2 ft. producing rotation



(*a*) (*b*)

FIG. 194.

around O in a positive direction. The other 100-lb. force at O, shown by a dotted line, causes translation exactly as if the original 100-lb. force had been applied at that point. The other forces being treated in the same way, the 200-lb. force is shown equivalent to a couple of moment $-4 \times 200 = -800$ ft.-lb. and a single force of 200 at O. Also, the 300 lb. is equivalent to a couple of moment $5 \times 300 = +1{,}500$ ft.-lb. The moment of the entire system is

$$\Sigma M = +200 - 800 + 1{,}500 = +900 \text{ ft.-lb.}$$

The resultant of the forces (dotted lines) at O is found as in Chap. IV, *viz.*,

$$\Sigma X = 100 \cos 40° + 200 \cos 10° + 300 \cos 90° = 273.6$$
$$\Sigma Y = 100 \sin 40° - 200 \sin 10° - 300 \sin 90° = -270.4$$
$$R = \sqrt{273.6^2 + 270.4^2} = 384.7 \text{ lb.}$$
$$\theta = \tan^{-1}\left(-\frac{270.4}{273.6}\right) = 135°20'$$

The resultant is then a force of 384.7 lb., making an angle of 135°20′ with the x axis. It is directed to the right and down. It must have a moment of 900 ft.-lb. Then

$$384.7a = 900$$
$$a = 2.34$$

Then R must be drawn to the right and down. Since the moment is plus, R must cause counterclockwise rotation. Its location is as shown in Fig. 194. The resultant is then a force and a couple.

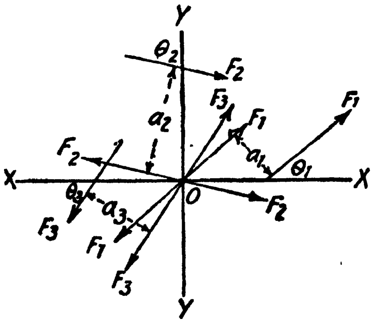


FIG. 195.

93. General Method.—Now, take a set of forces F_1, F_2, F_3, etc., at the distances a_1, a_2, a_3, etc., respectively, from the origin O in Fig. 195. At O, insert two forces F_1 parallel to F_1, and opposite in direction. Two of the forces form a couple of moment a_1F_1 and a single force F_1 at 0. Doing the same with F_2, F_3, etc., we obtain a set of couples tending to rotate around O, the resultant moment of which is

$$\Sigma M = F_1a_1 + F_2a_2 + F_3a_3 + \cdots = \Sigma Fa$$

There is also a set of concurrent forces at O, the resultant of which is

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$$

and

$$\theta = \tan^{-}\left(\frac{\Sigma Y}{\Sigma X}\right)$$

The resultant R is located by drawing a vector, making an angle θ with the x axis at a distance a from the origin so that $Ra = \Sigma Fa$. The resultant, therefore, is equivalent to a force and a couple. If the system of forces is in equilibrium, $R = 0$. But this result can be true only when $\Sigma X = 0$ and $\Sigma Y = 0$. This condition shows that a body at O would not be translated by the system of forces. But, for equilibrium, there must be no rotation, a state that can obtain when, and only when, $\Sigma M = 0$. There are then three equations of condition for the solution of forces in equilibrium. If the original forces F_1, F_2, F_3, etc., had been resolved into their X and Y components, in the position in which they are located, there would then have been two systems of forces, one parallel to the X axis and the other parallel to the Y axis. But, from Chap. V, the resultant of a set of parallel forces is their algebraic sum. These sums are ΣX and ΣY, just as if they had been resolved into their components at point O and their sum taken.

If $\Sigma X = 0$ and $\Sigma Y = 0$ while ΣM does not equal 0, the resultant of the system of forces is a couple; also, if $\Sigma M = 0$ while ΣX and ΣY are not equal to 0, the resultant is a single force. But when all three of the foregoing expressions equal zero, the system of forces is in equilibrium.

94. Graphical Method.—Let F_1, F_2, and F_3 of Fig. 196 be three nonconcurrent forces. To find their resultant graphically, produce the lines of F_1 and F_2 to intersect at A. Lay off F_1' and F_2' equal, respectively, to F_1 and F_2. Complete the parallelogram. The diagonal R_1 is the resultant of F_1 and F_2. Now produce R_1 to intersect the line of F_3 at B. Lay off R_1' and BC or $F_3' = F_3$. Complete the parallelogram. The diagonal R_2 is the resultant of R_1 and F_3 or of F_1, F_2, and F_3. With reference to a body at O, not on R_2, the resultant is equivalent to a single force R_2 at O and a couple of moment aR_2, causing rotation about O. If O is on R_2, the resultant is a single force R_2, the moment of the couple being zero, since the moment arm is zero. If R_1 should be equal and parallel to F_3 but opposite in direction, the resultant force is zero, but there is a resultant couple of moment bR_2, where b is the distance between the parallel forces. If R_1 is equal to F_3 and coincides with it, then the resultant force and couple are each equal to zero. The system of forces is then in equilibrium.

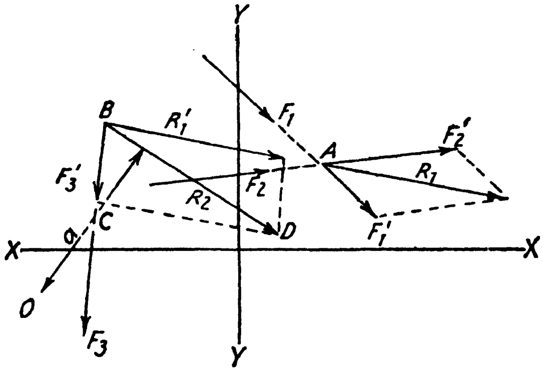


FIG. 196.

The preceding method of determining a resultant is general and may be extended to include any number of forces.

95. Multiple-force Body.—When a body is acted upon by forces at three or more points, it is called a *multiple-*

force body. In Fig. 197, the beam AC, hinged or fastened by a smooth pin at the wall A, is held horizontal by the tie rod BD and carries a weight W at C. It is then a multiple-force body.

Now, set AC out as a free body. At C the weight W acts down. At B there is the action of a two-force piece BD. Its line of action is that of BD. Since AC is in equilibrium under the action of three forces, they must meet in a common point. The directions of W and T are fixed, thus locating point E through which the force A must pass. Figure 197*b* shows that the external reaction at A is not in the direction of AC, as would be the case in a two-force piece. The stress in a multiple-force piece cannot be determined by methods thus far developed.

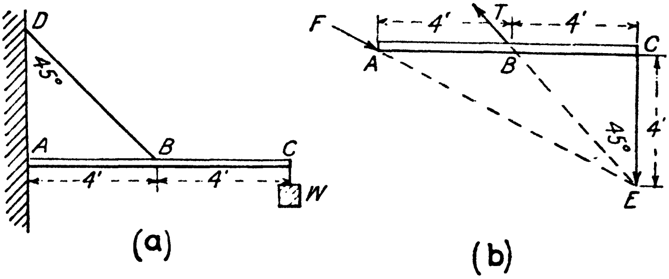


FIG. 197.

96. Hinge and Pin Reactions. *Graphical Method.*—Let $W = 2{,}000$ lb. Since at E there are three concurrent forces in equilibrium, they may be taken as the sides of a force triangle (see Fig. 198). If the force triangle is accurately drawn, the forces T and F may be found by measurement. They may also be found by solving the triangle. From Fig. 197*b*, BCE is seen to be an isosceles right triangle. Then $CE = 4$. Tan $\angle AEC = \frac{8}{4} = 2$, and $\angle AEC = 63°26'$. Then the angles of the force triangle are shown in the figure.

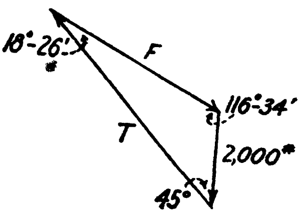


FIG. 198.

$$\frac{T}{2{,}000} = \frac{\sin 116°34'}{\sin 18°26'}$$
$$T = 5{,}658 \text{ lb.}$$
$$\frac{F}{2{,}000} = \frac{\sin 45°}{\sin 18°26'}$$
$$F = 4{,}472 \text{ lb.}$$

97. Algebraic Solution.—If the student has access to a drawing board and good instruments, so that accurate drawings may be made, the graphical solution will always give satisfactory results. However, since such conditions are not always found, another method of solution will be shown, called the "algebraic," which is good for all cases.

In Chap. IV, it was shown that a force could be replaced by two components at right angles to each other. Whenever a force is unknown both in magnitude and direction, it will in general be replaced by its horizontal and vertical components. The horizontal component will be designated by joining a subscript H to the letter at the point through which the force acts; the vertical component will be similarly indicated by a subscript V. In either case there are two unknowns, but the use of two components is a convenience in forming ΣX and ΣY. Figure 199*a* shows AC as a free body. Force F is replaced by A_H and A_V. One thing that confuses the student at this point is the direction to give the components. In many cases, it is possible to determine the directions by inspection. For example, under the action of the load and the tie rod T, there is a tendency to move AC to the left. The pin reacts on AC to prevent this motion. That is, A_H acts to the right. It is not always possible to find the direction of A_V by inspection. However, in any case, the directions may be assigned at random. If the proper directions have been assigned, the solution of the resulting equations will give positive values for the unknowns. But if the assumed directions are wrong, the value of the unknowns will be found with negative signs.

Since the forces are in equilibrium,

$$\begin{aligned}\Sigma X &= A_H - T\cos 45^\circ = 0\\ \Sigma Y &= A_V + T\sin 45^\circ - 2{,}000 = 0\\ \Sigma M_B &= -4A_V - 4\times 2{,}000 = 0\end{aligned}$$

Solving these equations, we have

$$A_V = -2{,}000\text{ lb.},\qquad T = 5{,}656\text{ lb.},\qquad A_H = 4{,}000\text{ lb.}$$

Since A_V is negative, it shows that the wrong direction was assumed. It therefore acts down. F may now be found from its components as follows:

$$\begin{aligned}F &= \sqrt{4{,}000^2 + 2{,}000^2} = 4{,}472\text{ lb.}\\ \theta &= \tan^{-1}\left(\tfrac{1}{2}\right) = 26^\circ 34'\end{aligned}$$

The solution of problems can be simplified by a judicious choice of the moment center in forming $\Sigma M = 0$. Always choose a point, if possible, where two or more unknowns intersect. This procedure often leaves an equation with one unknown, which may be found at once.

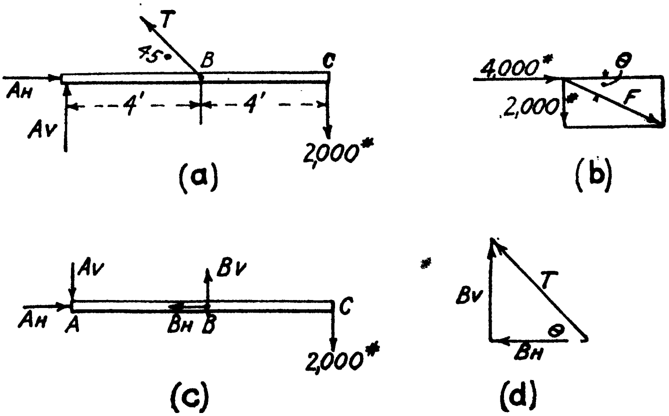


FIG. 199.

In Fig. 199*a*, either points A or B could be used. For example,

$$\begin{aligned}\Sigma M_A &= 4\cos 45^\circ \times T - 8\times 2{,}000 = 0\\ T &= 5{,}656\end{aligned}$$

In solving this problem, force T, known in direction because of being the action of a two-force piece, was not replaced by its components. However, the solution will now be given with these components introduced. Again, set AC out as a free body (Fig. 199*c*).

$$\begin{aligned}\Sigma M_A &= 4B_V - 8 \times 2{,}000 = 0\\ B_V &= 4{,}000 \text{ lb.}\\ \Sigma Y &= -A_V + 4{,}000 - 2{,}000 = 0\\ A_V &= -2{,}000 \text{ lb.}\\ \Sigma X &= A_H - B_H = 0\end{aligned}$$

Since there are four unknowns and only three equations, a fourth must be found before the solution can be completed. This is determined as follows:

Let T make an angle θ with the horizontal (Fig. 199*d*). Then

$$B_V = B_H \tan \theta$$

In the example,

$$\theta = 45^\circ$$

$$\therefore B_H = B_V = 4{,}000 \text{ lb.} = A_H$$

$$T = \sqrt{\overline{4{,}000}^2 + \overline{4{,}000}^2} = 5.656 \text{ lb.}$$

98. Illustrations.—Some examples will now be solved to show the application of the preceding principles to problems of mechanics. The following steps are suggested as a guide to the student and will be followed when possible in the problems solved.

a. Set out the entire structure as a free body, and solve for the external reactions if possible.

b. Classify each member of the body as to multiple- and two-force pieces.

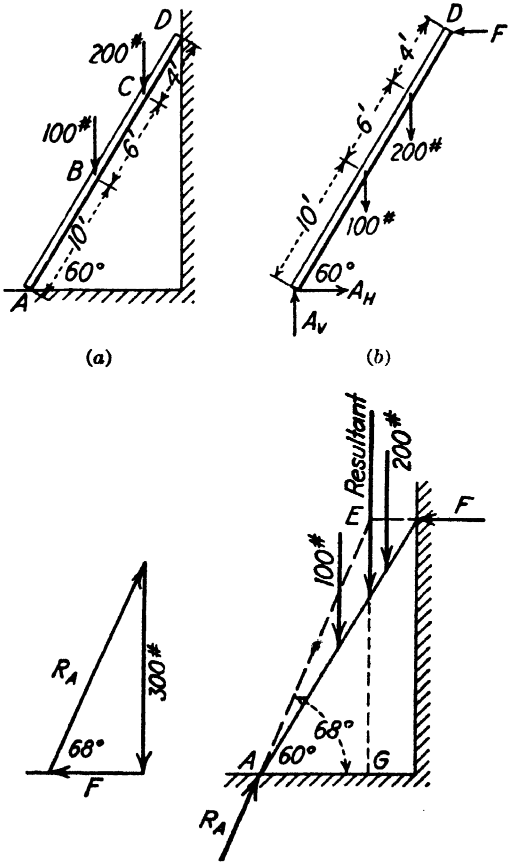
c. If there are any points where only two-force pieces meet, set such points out as free bodies.

d. Set multiple-force pieces out as free bodies.

e. By setting up $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$ for the free-body diagrams, in general enough equations will be found for the determination of the unknowns.

Example 1.—A beam 20 ft. long weighing 100 lb. is pinned to the floor at *A* and rests against a smooth wall at *D*. It carries a load of 200 lb. at *C*. Find the reaction of *A* and *D* (Fig. 200*a*).

Solution.—There is but one member in the structure, *viz.*, *AD*. Since forces act on it at more than two points, it is a multiple-force piece. Now, set *AD* out as a free body. Since the wall is perfectly smooth, the wall reaction at *D* will be normal to the wall. The reaction at *A* being unknown in magnitude and direction, will be replaced by its two components A_V and A_H. The weight of the beam is at *B* and the 200 lb. at *C* (see Fig. 200*b*). In order to eliminate two of the unknowns, first take moments around *A*. Then



(*a*) (*b*)

(*c*)

Fig. 200.

$$\Sigma M_A = 100 \times 10 \cos 60° + 200 \times 16 \cos 60° - F \times 20 \sin 60° = 0$$
$$F = 121 \text{ lb.}$$
$$\Sigma X = A_H - F = 0$$
$$A_H = 121 \text{ lb.}$$
$$\Sigma Y = A_V - 100 - 200 = 0$$
$$A_V = 300 \text{ lb.}$$

Alternate Method.—The resultant of the 100-lb. and 200-lb. loads is 300 lb. acting on the beam at 4 ft. from *B*. The beam may be considered as a body held in equilibrium by three concurrent forces. The point of concurrence is at *E*, of which the position is determined by producing the line of action of the resultant load of 300 lb. to an intersection with the line of action of the force *F*. The direction of R_A is now determined by connecting point *A* with point *E*.

In Fig. 200*c*, length *AG* is equal to 14 cos 60° = 7 ft., and length *EG* is equal to 20 sin 60° = 17.32 ft.

$$\tan \angle EAG = \frac{17.32}{7} = 2.4743$$
$$\angle EAG = 68°$$

Solving the force triangle, we have

$$R_A = \frac{300}{\sin 68°} = \frac{300}{0.9272} = 323.5 \text{ lb.}$$

Example 2.—An *A* frame carries a load of 1,000 lb. as shown in Fig. 201. Find the floor reactions at *A* and *E* and the pin reactions at *B*, *C*, and *D*.

Solution.—Following the steps suggested in the preceding articles, first set the entire *A* frame out as a free body (Fig. 201). Since the load or action is vertical, the reactions at *A* and *E* will also be vertical, there being no tendency for the frame to move sidewise.

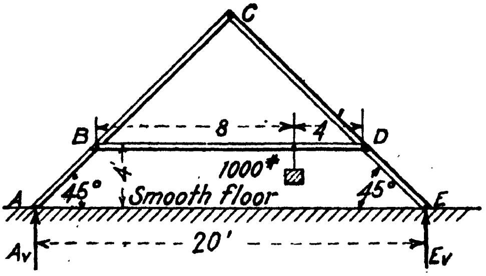


FIG. 201.

$$\Sigma M_A = -12 \times 1{,}000 + 20 \times E_v = 0$$
$$E_v = 600 \text{ lb.}$$
$$\Sigma Y = A_v + 600 - 1{,}000 = 0$$
$$A_v = 400 \text{ lb.}$$

All the members of the A frame are multiple-force pieces, being acted upon by forces at more than two points. Now, set all three of them out as free bodies (Fig. 201*a*). Since all the pin reactions are unknown, they must be represented by their horizontal and vertical components. The student should note very carefully the directions of these components. Since the member BD pulls downward on AC and also prevents the motion of AC to the left, B_H acts to the right on AC and B_V acts downward. Since AC prevents BD from falling and also exerts a pull to the left on BD in the free-body diagram of BD, B_H acts to the left and B_V acts upward. A similar condition is shown at C and D. This result verifies the principle that action and reaction are equal; and when members of the A frame are joined together, there must be equilibrium at each of the points B, C, and D. That is, for each of the pins, $\Sigma X = 0$ and $\Sigma Y = 0$. Referring again to the free-body drawing of AC, the student will observe that there are four unknowns. Since only three equations are possible, these unknowns cannot be found without further conditions.

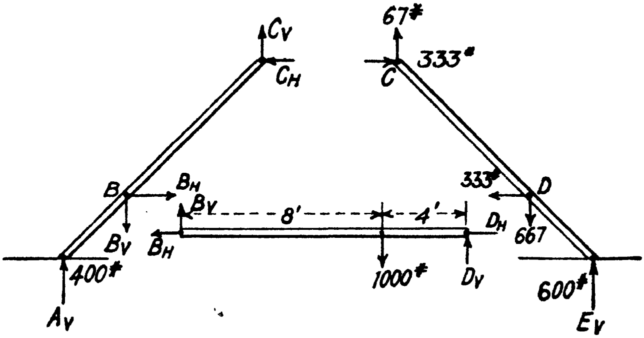
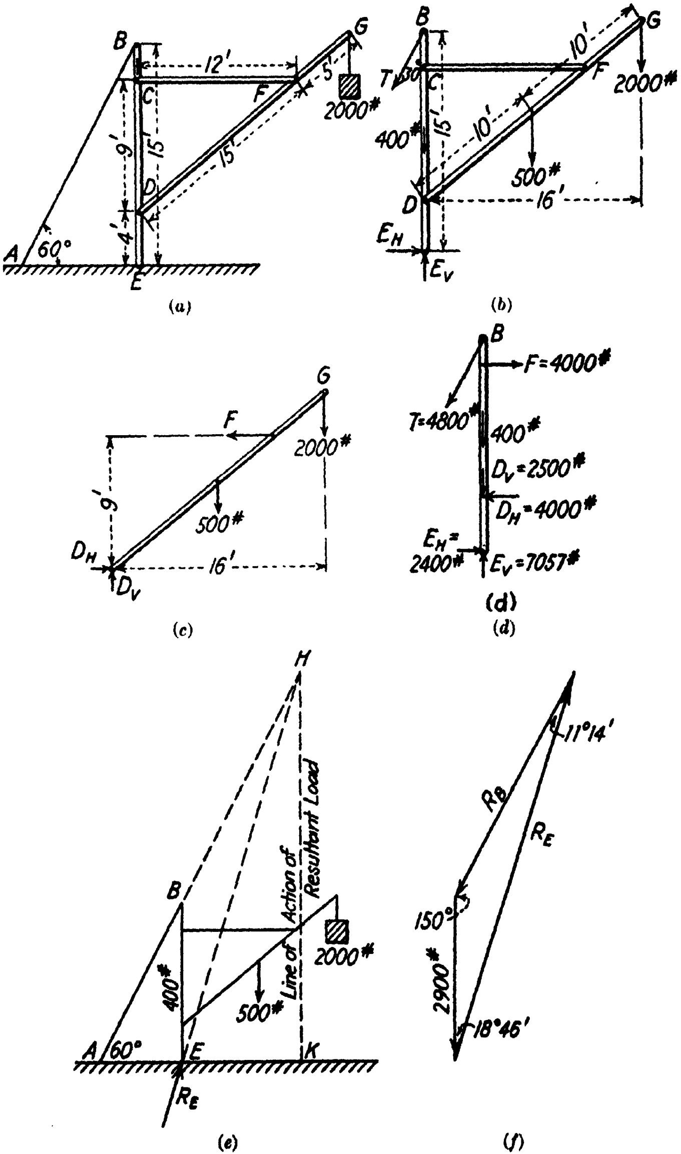


FIG. 201*a*.

Now, consider the free-body drawing of BD. From this sketch, it is possible to determine the two vertical components by applying $\Sigma M = 0$ and $\Sigma Y = 0$.

$$\begin{aligned} \Sigma M_B &= -8 \times 1{,}000 + 12 \times D_V = 0 \\ D_V &= 667 \text{ lb.} \\ \Sigma Y &= B_V - 1{,}000 + 667 = 0 \\ B_V &= 333 \text{ lb.} \end{aligned}$$

Inserting this value of B_V on the free-body drawing of AC, we have but three unknowns remaining and the solution may be completed. By taking moments around C, two of the three remaining unknowns are eliminated.



(a) (b)

(c) (d)

(e) (f)

FIG. 202(a–f).

$$\begin{aligned}
\Sigma M_C &= -10 \times 400 + 333 \times 6 + 6 \times B_H = 0 \\
B_H &= 333 \text{ lb.} \\
\Sigma X &= 333 - C_H = 0 \\
C_H &= 333 \text{ lb.} \\
\Sigma Y &= 400 - 333 + C_V = 0 \\
C_V &= -67 \text{ lb.}
\end{aligned}$$

The minus sign indicates that the wrong direction was assigned to C_V. It acts downward. From the free-body drawing of BD, it is evident that $D_H = B_H = 333$. As a check on the solution, insert the known values of the free-body drawing of CE (Fig. 201*a*).

$$\begin{aligned}
\Sigma X &= 333 - 333 = 0 \\
\Sigma Y &= +600 - 667 + 67 = 0 \\
\Sigma M_E &= 4 \times 667 + 4 \times 333 - 333 \times 10 - 10 \times 67 = 0
\end{aligned}$$

The resultant pin reaction is found as follows:

$$B = \sqrt{B_H^2 + B_V^2} = \sqrt{(333)^2 + (333)^2} = 471 \text{ lb.}$$
$$\theta = \tan^{-1}\left(\tfrac{333}{333}\right) = 45^\circ$$

In the same way, the pin reactions at C and D may be found.

Example 3.—Figure 202*a* shows a derrick carrying a load of 2,000 lb. The boom DG weighs 500 lb., and the mast weighs 400 lb. Find the pin reactions at E, D, C, B, and F.

Solution.—First, set the entire derrick out as a free body (Fig. 202*b*). The derrick is acted upon by the load, the weights of the members, the cable AB, and the ground at E. Since AB is a two-force piece, the direction of its action is known. Call this force T. The reaction of E is unknown, and therefore we represent it by the components E_V and E_H.

Taking moments, we have

$$\begin{aligned}
\Sigma M_B &= 15E_H - 8 \times 500 - 16 \times 2{,}000 = 0 \\
E_H &= 2{,}400 \\
\Sigma X &= 2{,}400 - T \sin 30^\circ = 0 \\
T &= 4{,}800 \text{ lb.} \\
\Sigma Y &= E_V - 400 - 500 - 2{,}000 - 4{,}800 \cos 30^\circ = 0 \\
E_V &= 7{,}057 \text{ lb.}
\end{aligned}$$

Next, classify the members of the derrick. AB and CF are two-force pieces whereas EB and DG are multiple-force pieces. Since there is no point at which only two-force pieces meet, the next step in the solution will be to set out the multiple-force pieces as free bodies (Figs. 202*c* and 202*d*). Applying the equations of condition to Fig. 202*c*, there result

$$\begin{aligned}
\Sigma M_D &= 9F - 8 \times 500 - 16 \times 2{,}000 = 0 \\
F &= 4{,}000 \text{ lb.} \\
\Sigma X &= D_H - 4{,}000 = 0 \\
D_H &= 4{,}000 \text{ lb.} \\
\Sigma Y &= D_V - 500 - 2{,}000 = 0 \\
D_V &= 2{,}500 \text{ lb.}
\end{aligned}$$

The student can now find the resultant pin reactions by the method shown in the preceding problem.

Check.—Since the forces are all known, the free-body drawing of *BE* may now be used as a check on the solution.

$$\begin{aligned}
\Sigma X &= 2{,}400 - 4{,}000 + 4{,}000 - 4{,}800 \sin 30^\circ = 0 \\
\Sigma Y &= 7{,}057 - 2{,}500 - 400 - 4{,}800 \cos 30^\circ = 0 \\
\Sigma M_B &= 15 \times 2{,}400 - 11 \times 4{,}000 + 2 \times 4{,}000 = 0
\end{aligned}$$

Alternate Method.—By means of the principle of concurrence, the reactions at *B* and *E* can be found. The direction of the reaction at *B* is known to be along the line *BA*.

By taking moments about *D*, the position of the resultant of the three loads can be located thus:

$$2{,}900x = 500 \times 10 \sin \angle CDF + 2{,}000 \times 20 \sin \angle CDF$$

$$\sin \angle CDF = \tfrac{12}{15} = 0.8$$

Then

$$2{,}900x = 500 \times 10 \times 0.8 + 2{,}000 \times 20 \times 0.8$$

$$x = \frac{40 + 320}{29} = \frac{360}{29} = 12.41 \text{ ft.} = EK$$

The line of action of the resultant is 12.41 ft. to the right of the line *EB*.

Figure 202*e* shows the resultant in position with the point of concurrence at *H*. The direction of the reaction at *E* is now known.

R makes an angle *EHK* with the vertical.

$$\tan \angle EHK = \frac{EK}{HK}$$

where *HK* equals 15 + 12.41 tan 60°

$$\tan \angle EHK = \frac{12.41}{36.49} = 0.34$$

$$\angle EHK = 18^\circ 46'$$

Figure 202*f* shows the force triangle for the forces acting on the derrick as a whole. By the law of sines,

$$\frac{R_E}{2{,}900} = \frac{\sin 150°}{\sin 11°14'} = \frac{0.50}{0.1948}$$
$$R_E = \frac{0.50}{0.1948} \times 2{,}900 = 7{,}443 \text{ lb.}$$
$$\frac{R_B}{2{,}900} = \frac{\sin 18°46'}{\sin 11°14'} = \frac{0.3217}{0.1948}$$
$$R_B = \frac{0.3217}{0.1948} \times 2{,}900 = 4{,}798 \text{ lb.}$$

But R_B is identical with the stress in BA.

Example 4.—A cantelever truss, shown in Fig. 203*a*, carries a load of 1,200 lb. Find the pin reactions at A, B, C, and E.

Solution.—First, set the entire structure out as a free body (Fig. 203*b*). Since the member AD is a two-force piece, the wall reaction at A is horizontal. At B the action is caused by a multiple-force piece; therefore the reaction will be unknown in magnitude and direction and is represented by B_H and B_V. Then

$$\begin{aligned} \Sigma M_B &= 2P - 10 \times 1{,}200 = 0 \\ P &= 6{,}000 \text{ lb.} \\ \Sigma X &= 6{,}000 \text{ lb.} - B_H = 0 \\ B_H &= 6{,}000 \text{ lb.} \\ \Sigma Y &= B_V - 1{,}200 = 0 \\ B_V &= 1{,}200 \text{ lb.} \end{aligned}$$

AD and CD are two-force pieces, whereas BE and DF are multiple-force pieces. Since there are no points at which only two-force pieces meet, the next step in the solution will be to set out the multiple-force pieces as free bodies (Figs. 203*c* and *d*). In the free-body drawing of BE, since CD is a two-force piece, its line of action is known. At E, the action is caused by a multiple-force piece, and the reaction must be represented by its components E_V and E_H. Applying the equations of condition,

$$\begin{aligned} \Sigma M_C &= -1{,}200 \times 4 + 4 \times E_V = 0 \\ E_V &= 1{,}200 \text{ lb.} \\ \Sigma Y &= -F_1 \cos 45° + 1{,}200 + 1{,}200 = 0 \\ F_1 &= 3{,}394 \text{ lb.} \\ \Sigma X &= -6{,}000 + 3{,}394 \cos 45° + E_H = 0 \\ E_H &= 3{,}600 \text{ lb.} \end{aligned}$$

The student may now apply the equations of condition to the free-body drawing of DF to check the work, also to calculate the resultant pin reaction.

Example 5.—Figure 204*a* shows a truss supported at A and B and carrying a load of 4,000 lb. Find the external reactions and the stresses in all the members, assuming that all joints are pin-connected.

Solution.—First, set the entire truss out as a free body. The direction of the reaction at A is known since AF is a two-force piece.

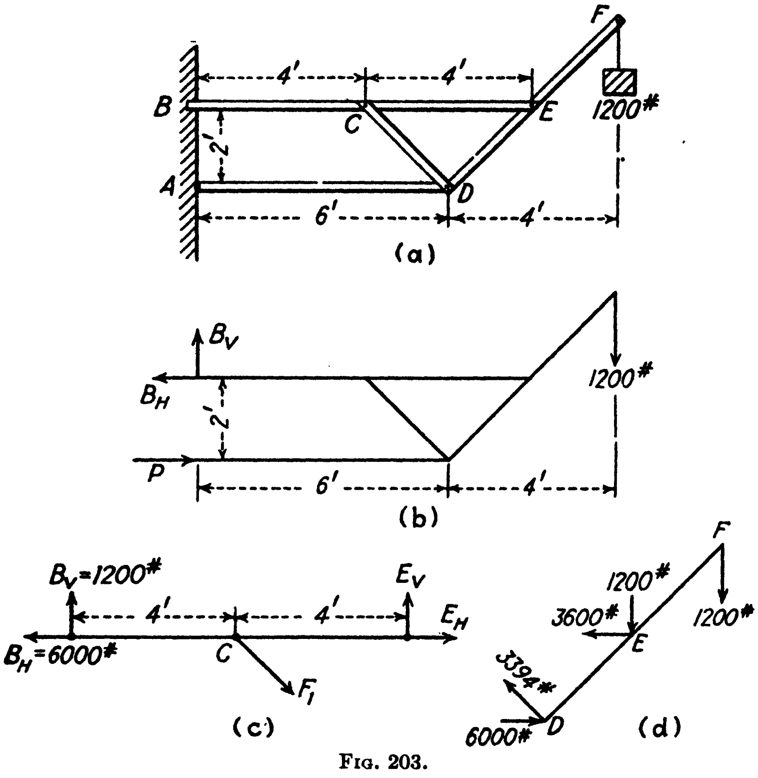


FIG. 203.

At B, the reaction being unknown, it is represented by B_H and B_V. Then

$$
\begin{aligned}
\Sigma M_A &= 9B_V - 4{,}000 \times 19 = 0 \\
B_V &= 8{,}444 \text{ lb.} \\
\Sigma Y &= -P \times \tfrac{4}{5} + 8{,}444 - 4{,}000 = 0 \\
P &= 5{,}555 \text{ lb.} \\
\Sigma X &= B_H - \tfrac{3}{5} \times 5{,}555 = 0 \\
B_H &= 3{,}333 \text{ lb.}
\end{aligned}
$$

Since all the members of the truss are two-force pieces, the stresses in the members may be found by setting out points as free bodies. This is left as an exercise for the student. A method known as the "section method" will be here introduced.

Suppose, for example, that the stresses in *FE*, *EB*, and *BC* are to be found. Cut the truss into two parts by a section through the three members of which stresses are to be found. Either portion of the truss may now be set out as a free body. Suppose that *CDE* is taken as in Fig. 204*b*. The unknown forces acting on *CDE* are F_1, F_2, and F_3. The known load of 4,000 is distributed at *C* and *D* in the ratio of 1:2, by the theorem of parallel forces. First, take moments around *E* in order that two of the unknowns may be eliminated. Then

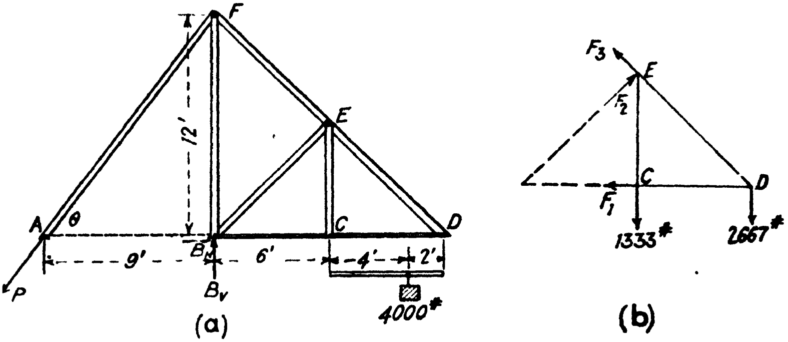


FIG. 204.

$$\Sigma M_E = -6F_1 - 6 \times 2{,}667 = 0$$
$$F_1 = -2{,}667 \text{ lb.}$$

This result shows that F_1 is in compression. Next, take moments around point *D* to eliminate F_1 and F_3.

$$\Sigma M_D = 8.5F_2 + 6 \times 1{,}333 = 0$$
$$F_2 = -941 \text{ lb.}$$

That is, F_2 is in compression.

$$\Sigma M_B = 8.5F_3 - 6 \times 1{,}333 - 12 \times 2{,}667 = 0$$
$$F_3 = 4{,}706 \text{ lb. tension}$$

The section method can be applied if not more than three unknown stresses are involved in dividing the truss into two parts. Though it may be used in finding the stresses in all the members, its advantage lies in finding certain stresses independent of the other stresses in the truss. The student will find it to his advantage to apply both the point and the section method.

Problems

179. A timber of uniform cross section, weighing 200 lb., is hinged at its lower end and held at an angle of 60° with the horizontal by a rod attached as shown in Fig. 205. A cylinder of which the weight is 60 lb. is placed between the timber and the wall. What are the horizontal and vertical components of the reaction at *A*? What is the pressure between the cylinder and the timber?

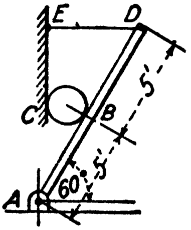


Fig. 205.

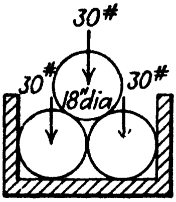


Fig. 206.

180. Three smooth cylinders each 18 in. in diameter and each weighing 30 lb. are placed in a box as shown in Fig. 206. What are the pressures at all points of contact?

181. Frequently the height of the water on top of a dam is raised by means of a flashboard as shown in Fig. 207. What horizontal force at *A* will tip the board? What is the pressure against the stop at *B*? The length of the flashboard is 10 ft.

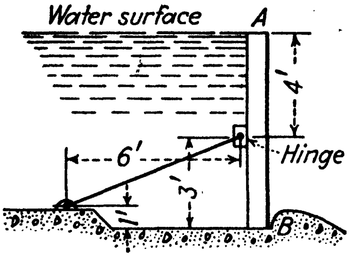


Fig. 207.

182. The *A* frame in Fig. 201 has an additional load of 1,200 lb. at the center of *BD*. All members weigh 90 lb. per ft. of length. What are the pin reactions at *B*, *C*, and *D*?

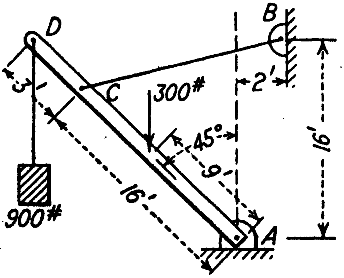


Fig. 208.

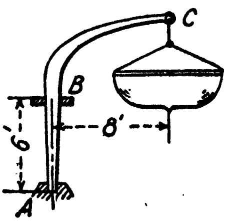


Fig. 209.

183. Figure 208 shows a wooden brace hinged to a floor at *A* and held by means of the rod *BC*. The weight of the brace is 300 lb. and may be assumed to be applied as shown. What will be the horizontal and vertical components of the pressure at *A*? What is the stress in *BC*?

184. An ocean liner has an arrangement for supporting lifeboats and for lowering them over the side as shown in Fig. 209. There is a socket at *A* and a smooth hole through the deck rail at *B*. If the boat and its load weigh 2,000 lb., what are the reactions at *A* and *B*?

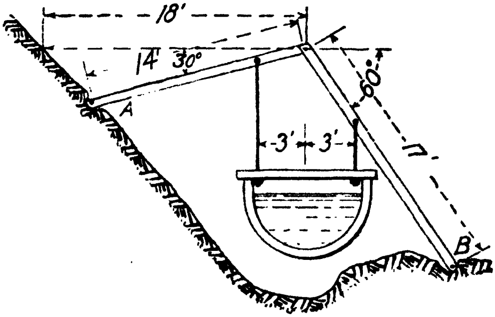


FIG. 210.

185. In an irrigation project, it was found necessary to cross low ground or else swing the canal to the left by cutting into the solid rock. It was decided to run the canal as a flume and support it on a number of frames as shown in Fig. 210. The two members rest in sockets in solid rock at points *A* and *B*. These sockets may be considered as hinges. What will be the vertical and horizontal components of the reactions at *A* and *B*? The weight of the water in the flume supported by each frame is estimated as 18,200 lb.

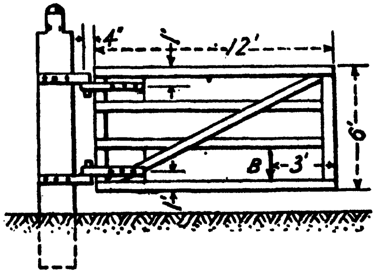


FIG. 211.

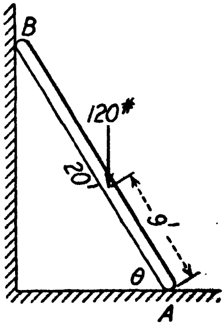


FIG. 212.

186. A gate has a weight of 300 lb., which may be considered as uniformly distributed (Fig. 211). A small boy weighing 90 lb. climbs up on the gate at the point *B*. What will be the reactions at the hinges?

187. A timber weighing 120 lb. is supported by the floor and wall as shown in Fig. 212. If the coefficient of friction at the surfaces of contact is 0.4, at what angle with the horizontal will the timber be just on the point of sliding?

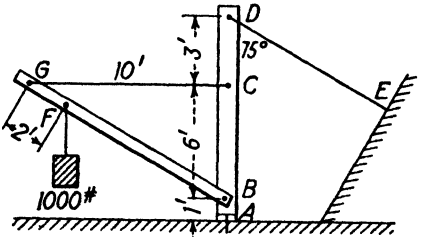


FIG. 213.

188. In the crane shown in Fig. 213, the member *BG* weighs 600 lb. which is considered to act midway between *B* and *F*. If other weights of members are neglected, what will be the stresses in the two tension rods? What will be the horizontal and vertical components of the reactions at *A* and *B*?

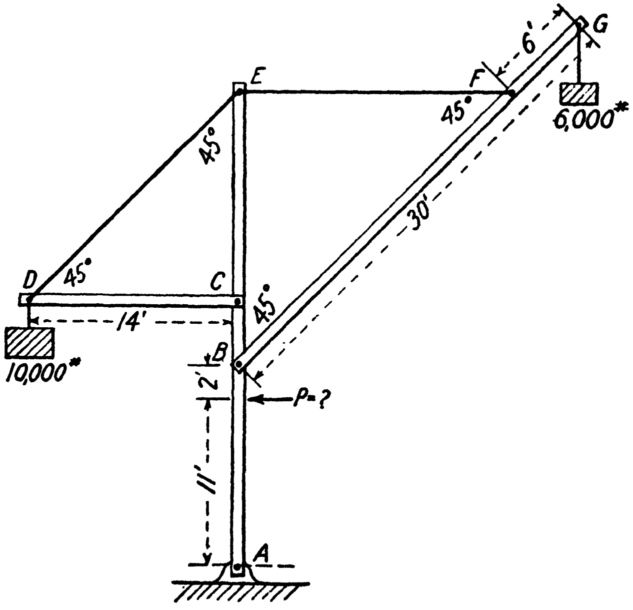


FIG. 214.

189. What force *P* is needed to prevent the framework in Fig. 214 from overturning? Find stresses and pin reactions.

190. A brace is hinged at one end to a vertical wall and at the other end to a beam 12 ft. long and weighing 400 lb. and which is also hinged to a vertical wall as shown in Fig. 215. The beam carries a load of 600 lb. at the free end. What will be the compressive stress in the brace, and what will be the values of the vertical and horizontal components of the reaction at hinge A?

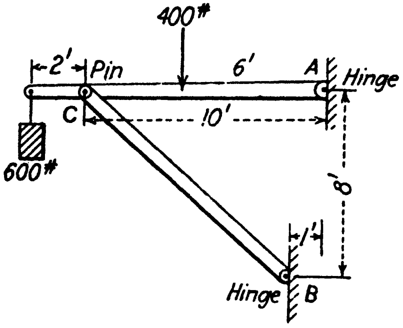


FIG. 215.

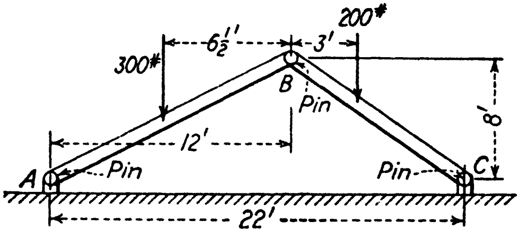


FIG. 216.

191. The jointed frame in Fig. 216 carries loads as shown. What are the vertical and horizontal components of the pin reactions?

192. A piece of uniform cross section weighing 150 lb. is hinged at its upper end to a horizontal beam and is held in the position shown in Fig. 217 by means of the horizontal brace BD. What will be the reaction at C and the stress in BD?

193. In an iron foundry a jib crane, shown in Fig. 218, was used to handle the molten ore. With the moving load in the position shown, what will be the stresses in the members A, B, and C? What is the horizontal component of the reaction at the upper end of the vertical member?

194. The top of a ladder rests against a smooth vertical wall at a point 30 ft. above the ground. The foot of the ladder is placed

15 ft. horizontally from the base of the wall. The ladder, of which the center of gravity is 10 ft. above the ground, weighs 50 lb. The coefficient of friction between the ladder and the ground is 0.3. Find the vertical height to which a man weighing 180 lb. may ascend before the ladder begins to slip.

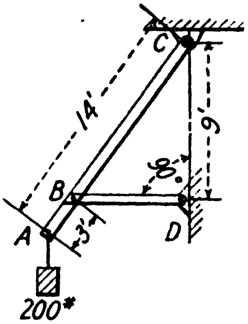


Fig. 217.

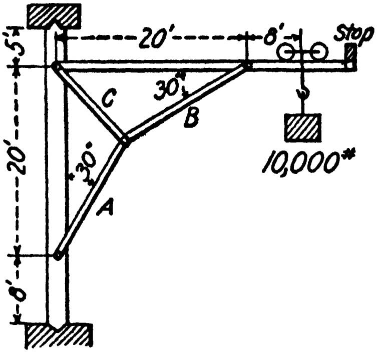


Fig. 218.

195. In the previous problem, if the man ascends to the top of the ladder, what must be the horizontal force at the foot of the ladder to prevent slipping?

CHAPTER IX

FRAMED STRUCTURES—ROOF AND BRIDGE TRUSSES

99. Framed Structure.—A framed structure is composed of two-force pieces, pin-connected at their ends so that the entire frame acts as a rigid body.

The points where two or more members meet are called *joints.* The members of the structure extend from one joint to another only. The loads carried by the structure will be considered as acting at the joints. If the weights of the members are neglected, each one is a tension or compression member. Then, at each joint of the framed structure, there are concurrent forces in equilibrium, all of which are known in direction.

The only geometrical figure that cannot change in shape and size when the length of the sides remains unchanged is made of two-force pieces, arranged so as to form triangles, and is called a *truss.*

A *roof* or *bridge truss* is a framed structure with the plane vertical, resting on supports.

The distance between supports is called the *span.*

The *rise* of a truss is the distance from the highest point to the horizontal line joining the supports of the truss.

The *pitch* is the ratio of the rise to the span. If h is the rise and l the span, the pitch $p = h/l$.

The *upper chord* of a truss is formed by the upper line of members from one support to the other through the apex.

The *lower chord* of a truss is formed by the lower member from one support to the other.

100. General Types.—Figures 219 to 225 show some of the most common types of roof truss.

If the truss of Fig. 219 is to carry a ceiling load, a vertical tie rod is added to relieve the bending stress at the center and prevent too great a deflection in the horizontal tie beam. This type of truss may be used for spans up to 20 ft.

If the span is over 20 ft., a type of truss shown in Fig. 221 may be used in which the upper chord members *AC* and *CE* are braced by the struts *BF* and *FD*. This truss is used for spans up to 35 ft. In case a ceiling or floor load is carried, the lower chord *AFE* should be braced by vertical tie rods from *B* and *D*.

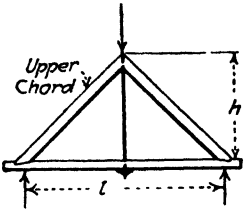


Fig. 219.

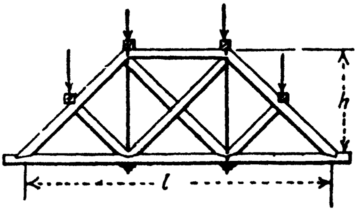


Fig. 220.

For spans from 30 to 45 ft., the queen rod truss shown in Fig. 220 may be used. If the load is the same on both sides of the truss, there will be no need of the inclined braces in the center panel. But, as this would not be the case when the wind blows, the braces should be inserted. If the span is from 40 to 50 ft., the horizontal members in the center panels would be from 13 to 17 ft. long and would require vertical rods for bracing as shown in Fig. 224.

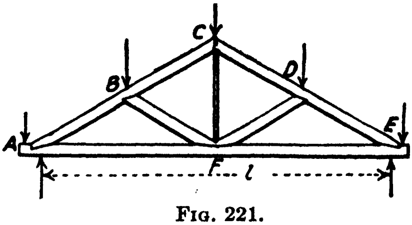


Fig. 221.

Figure 222 is a combination of types shown in Figs. 221 and 219. The number of panels depends on the span as well as other specifications. The span seldom exceeds 60 to 75 ft.

101. Definitions of Truss Parts.—The points *A*, *B*, *C*, *D*, *E*, and *F*, of Fig. 221, at which the members are pinned together are called *joints*. The live and dead loads are assumed as acting at the joints of the upper chords, whereas ceiling loads, if any, would act at the joints of the lower chord. The lengths *AB*, *BC*, *CD*, and *DE* are called *panels*.

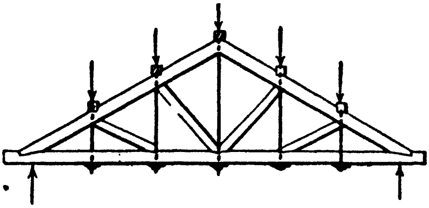


Fig. 222.

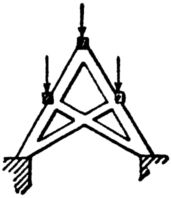


Fig. 223.

Some of the members *AB*, *BF*, etc., are in compression and are called *struts*, whereas others such as *CF*, *AF* are in tension and are called *ties*. Now, timber will resist crushing stresses to good advantage. But on account of the low shearing strength of timber, when a pulling or tensile stress is to be resisted a wooden member is often unsatisfactory. The wood in front of a pin or bolt will often shear out, leaving the rest of the member in perfect condition. Consequently, such a member as *CF* is sometimes made of steel. Trusses are often built up with wooden and steel members. The latter pass through the wood at the joint and are fastened by nuts.

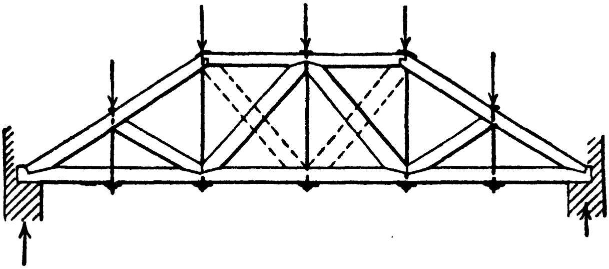


Fig. 224.

102. Best Support for Loads.—For the most economical design of trusses the loads should be applied at the joints by means of purlins or purlin trusses resting on *B*, *C*, and *D* of Fig. 225. If the purlin rests on the truss at a point between *A* and *B*, the effect of the load is to cause bending stress, in addition to compression. This requires a larger member in *AB*, for example, than if the load were at the joint. Under the assumption that the loads are carried in this manner, we shall show how the loads at the joints are figured.

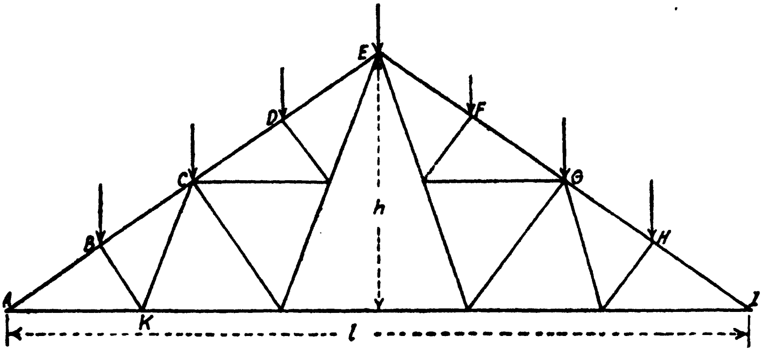


Fig. 225.

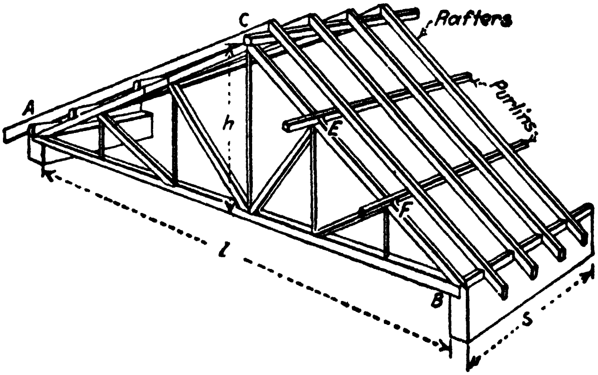


Fig. 226.

Figure 226 shows the section of a roof carried by the truss *ABC*. In practice, the truss should be under the center of the section; but, for a better view of the truss, in the figure it is placed at one side of the roof section. Suppose that the span $l = 45$ ft. and $h = 15$ ft. Then the pitch is $15 \div 45 = \frac{1}{3}$. Assume also that the trusses are 10 ft. apart and that the total weight carried is 50 lb. per sq. ft. of roof surface.

The length *BC* is

$$\sqrt{22.5^2 + 15^2} = \sqrt{731.25} = 27 \text{ ft.}$$

The total roof area carried by the truss is

$$2 \times 27 \times 10 = 540 \text{ sq. ft.}$$

The load is $540 \times 50 = 27{,}000$. Since there are six panels, each will carry

$$\frac{27{,}000}{6} = 4{,}500 \text{ lb.}$$

Now the figure shows that one-half of a panel load is carried by each of its end joints. Then the end joints *A* and *B* carry one-half a panel load, or 2,250 lb. All other joints such as *F*, *E*, *C* carry a full panel load, or 4,500 lb.

In general, if the total load is W and n is the number of panels, the load at the end joints is $W/2n$. For all other joints the load is W/n.

Example.—Suppose in Fig. 226 that $W = 60{,}000$ lb. and the number of panels $n = 6$. Find the joint loads.

Solution.—For the two end joints the load is

$$\frac{W}{2n} = \frac{60{,}000}{2 \times 6} = 5{,}000 \text{ lb.}$$

For all other joints,

$$\frac{W}{n} = \frac{60{,}000}{6} = 10{,}000 \text{ lb.}$$

103. Snow and Dead Loads.—The dead load is usually considered as made up of roof coverings, purlins, rafters, weight of trusses, and sometimes ceiling loads.

The snow load depends upon the locality and the pitch of the roof. Handbooks for architects and builders give detailed information on the figuring of dead and snow loads. It is not the purpose of this book to consider in detail such problems; rather, we shall assume a value in pounds per square foot of roof surface that for ordinary cases would be a maximum and then determine the stresses in the truss.

104. Wind Loads.—In the case of wind loads, the maximum pressure on the roof depends on the pitch. Experiments have shown that a horizontal wind with a velocity of 15 miles per hr. will produce a pressure of 1 lb. per sq. ft. on a vertical surface, 5 lb. for a 30-mile wind, 18 lb. for a 60-mile wind, and probably a pressure as high as 50 lb. per sq. ft. for a hurricane at a velocity of 100 miles per hr. For roof construction, engineers usually use a maximum of 40 lb. per sq. ft. Now, when the wind strikes a roof inclined to the vertical, not all of the force is exerted on the roof. Only the component perpendicular to the slope is used in finding the wind load.

Many formulas are used for determining the normal component of the wind load, but the straight-line formula is the one that will be given here because of its simplicity and ease of application. It is

$$P_n = \frac{PA}{45} \tag{31}$$

where P_n is the value of the wind pressure normal to the roof to use in figuring the wind load, P is the maximum wind pressure on a vertical surface and is taken as 40 lb., per sq. ft. and A is the number of degrees in the slope of the roof. If $A = 30°$ and $P = 40$ lb., then

$$P_n = \frac{40 \times 30}{45} = 27 \text{ lb., per sq. ft. approximately}$$

The following table gives values of P_n for slopes every 5° and also the angle of the roof in terms of the pitch.

dead load of 30 lb. per sq. ft. of roof surface, find the stresses in all the members of the truss.

106. Stresses Due to Wind Loads. *Algebraic Method of Computation.*—When a roof truss has forces acting on it at an angle with the vertical, such as a wind pressure, the truss would be moved sideways unless pinned at one or both ends. For trusses of short span, both ends are often fastened. But if the truss has a long span, and especially if it is made of iron, for bridge trusses, the custom is to pin one end and put the other end on rollers. This method of fastening would allow the truss to contract and expand under temperature changes without setting up the large temperature stresses that would otherwise result.

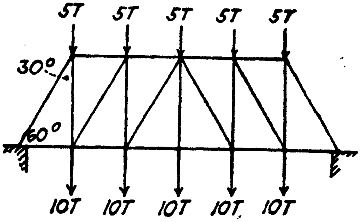


FIG. 230.

In dealing with wind pressures on a roof, it is only the component perpendicular to the roof that is considered.

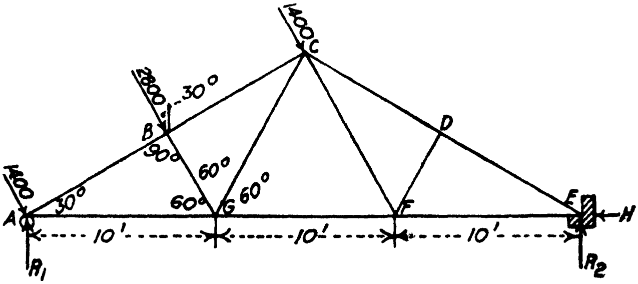


FIG. 231.

Example.—Find the stresses in the truss in Fig. 231 if the trusses are spaced 12 ft. apart.

Solution.—Since the left end of the truss is on rollers, the reaction at A is vertical. The reaction at E is unknown in magnitude and direction, and we then replace it by two components R_2 and H. H will balance the horizontal component of the wind. ABG is a 30° right triangle. Then $BG = \frac{1}{2}AG = 5$ ft.

$$AB = \sqrt{10^2 - 5^2} = \sqrt{75} = 8.66 \text{ ft.} = BC = CD, \text{ etc.}$$

The upper chord $AC = 17.32$ ft.

From the table of Art. 104 the normal component of the wind for a 30° inclination is 27 lb. per sq. ft. Then the total wind pressure on AC is

$$P = 17.32 \times 12 \times 27 = 5{,}600 \text{ lb. approximately}$$

Since there are two panels to carry the load, B takes one-half, or 2,800 lb., whereas A and C take one-fourth, or 1,400 lb.

To determine the reactions, take moments around A, the wind side of the truss

$$\Sigma M_A = 0$$
$$-2{,}800 \times 8.66 - 1{,}400 \times 17.32 + 30R_2 = 0$$
$$R_2 = 1{,}617 \text{ lb.}$$
$$\Sigma X = 0$$
$$5{,}600 \times \sin 30° - H = 0, \qquad H = 2{,}800 \text{ lb.}$$
$$\Sigma Y = 0$$
$$-5{,}600 \cos 30° + 1{,}617 + R_1 = 0$$
$$R_1 = 3{,}233 \text{ lb.}$$

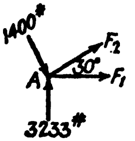


FIG. 231*a*.

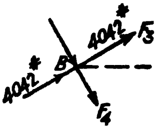


FIG. 231*b*.

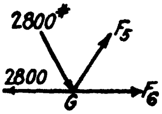


FIG. 231*c*.

a. Set joint A out as a free body. Let F_1 and F_2 be the stresses in AG and AB, respectively.

$$\Sigma Y = 0$$
$$-1{,}400 \cos 30° + F_2 \sin 30° + 3{,}233 = 0$$
$$F_2 = -4{,}042 \text{ lb.}$$
$$\therefore F_2 = 4{,}042 \text{ lb. compression}$$
$$\Sigma X = 0$$
$$1{,}400 \sin 30° - 4{,}042 \cos 30° + F_1 = 0$$
$$F_1 = 2{,}800 \text{ lb. tension}$$

b. Set joint B out as a free body.

Since the forces are at right angles,

$$F_3 + 4{,}042 = 0, \qquad F_3 = -4{,}042$$

and

$$-2{,}800 - F_4 = 0, \qquad F_4 = -2{,}800$$

c. Joint *G.*

$$\Sigma Y = 0$$
$$-2{,}800 \cos 30° + F_5 \cos 30° = 0$$
$$\therefore F_5 = +2{,}800 \text{ lb.}$$
$$\Sigma X = 0$$
$$-2{,}800 + 2{,}800 \cos 60° + 2{,}800 \cos 60° + F_6 = 0$$
$$F_6 = 0$$

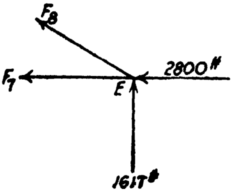


Fig. 231*d.*

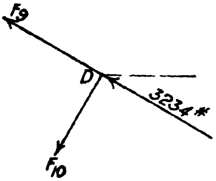


Fig. 231*e.*

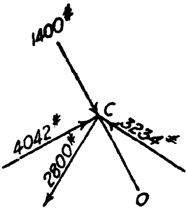


Fig. 231*f.*

d. Joint *E.*

$$\Sigma Y = 0$$
$$F_8 \sin 30° + 1{,}617 = 0$$
$$F_8 = -3{,}234 \text{ lb.}$$
$$\Sigma X = 0$$
$$-F_7 + 3{,}234 \cos 30° - 2{,}800 = 0$$
$$F_7 = -2{,}800 + 3{,}234 \cos 30° = 0$$

e. Joint *D.*

Since F_{10} is at right angles to the other forces,

$$F_{10} = 0, \qquad \text{and } F_9 = -3{,}234$$

If joint *F* were considered, *CF* would be found to equal zero. Check. A check on the work is obtained by considering joint *C.* Take $\Sigma X =$

$$4{,}042 \cos 30° - 3{,}234 \cos 30° - 2{,}800 \sin 30° + 1{,}400 \cos 60°$$
$$= 3{,}492 - 2{,}800 - 1{,}400 + 700$$
$$- 4{,}192 - 4{,}200 = -8 \text{ lb.}$$
$$\Sigma Y = -1{,}400 \sin 60° + 4{,}042 \sin 30° + 3{,}234 \sin 30° - 2{,}800 \cos 30°$$
$$= - 1{,}212.4 + 2{,}021 + 1{,}617 - 2{,}424.4 =$$
$$- 3{,}637.2 + 3{,}638 = 0.8 \text{ lb.}$$

ΣX and ΣY should equal zero, but on account of dropping decimals they were not quite zero. However, they were near enough to show the accuracy of the results. Several members were found equal to zero. This fact means that so far as wind load is concerned they are zero but the dead load would cause stresses.

When the dead load is figured for this same truss, the stresses for each result should be added to find the maximum stress.

The student should now figure the stresses with the wind pressure against the other side of the truss and get maximum stresses for dead and wind load. The greatest stress found in either case is the one to use in designing.

Member.	*AB*	*BC*	*CD*	*DE*	*EF*	*FG*	*GA*	*BG*	*GC*	*CF*	*FD*
Stress...	4042	4042	3234	3234	0	0	2800	2800	2800	0	0
Sign....	−	−	−	−			+	−	+		

107. Stresses in a Truss. *Graphical Method of Computation.*—In determining the stresses in a roof or bridge truss by graphical methods, the lettering of the drawing is an important item, and the method in common use will now be given. Figure 232*a* shows a simple roof truss carrying a load of 2,000 lb. R_1 and R_2 are the reactions at the supports. Since the load is in the middle and the truss is symmetrical, each reaction is one-half the load, or 1,000 lb. The three forces, *viz.*, 2,000, R_1 and R_2 divide the space outside the truss into three parts. These spaces are represented by small letters *a*, *b*, and *c*. Care should be taken to read the forces around the truss in the same direction, usually clockwise. The space within the truss is called *d*. Any force of the space diagram may then be

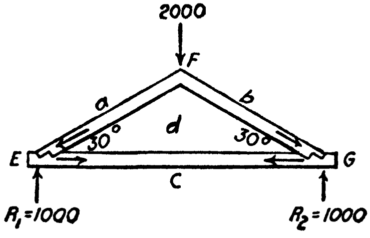


Fig. 232*a*.

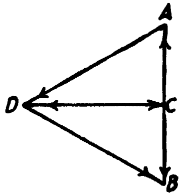


Fig. 232*b*.

represented by the letters of the two spaces that this force divides. Thus, the 2,000 lb. is read "force *ab*"; R_2 is read "force *bc*"; R_1 is read force "*ca*"; the stress in the horizontal member, "force *dc*"; etc.

We now construct the force diagram to determine the stresses in the members of the truss. Capital letters are used in the force diagram, corresponding to the small letters of the space drawing. Thus force *ab* in Fig. 232*a* would be *AB* in the force diagram, etc. In Fig. 232*b*, draw the vertical line *AB* 1 in. long to represent the 2,000 lb. Since the reactions are each 1,000 lb., R_2, or the force *bc*, shows as *BC*, with *C* at the center of *AB*. Also, force R_1, or *ca*, is shown as *CA*.

Now, at the joint *E*, there are three forces, *viz.*, R_1, the stress in *ad*, and that in *dc*. These forces are known to be in equilibrium, and will form a closed triangle. We proceed as follows: Begin with R_1, or force *ca*, which is known in magnitude and direction. In the force diagram, start at *C* and go to *A*, already drawn. From *A*, draw a line parallel to force *da*. It is drawn indefinite in length, as we know only its direction. Since we know that *ca*, *ad*, and *dc* will form a triangle, being balanced forces, we know that *dc* must end at *C*, the point from which we started. Also, *dc* is horizontal. Then, from *C*, draw a line horizontal, extending it until it intersects the line drawn from *A*. This intersection is the point *D*. We now put on the arrowheads on the triangle *CAD*, being careful to follow the same direction in which the forces were taken. Then, at the joint *E*, put the arrowheads on *ad* and *dc* pointing in the same directions as the arrowheads in triangle *CAD*. Since the arrowhead on *ad* points toward *E*, it shows that the force or stress in *ad* works against *E*. Member *ad* is then said to be in compression. Stress in *dc* works away from joint *E* and is in tension. *AD* is 1 in. long. But our scale was 2,000 lb. per in. Then there is a compression stress of 2,000 lb. in *ad*. *DC* scales 0.866 in. The stress in *dc* is 1,730 lb.

We go now to joint G. There are three forces, *viz.*, R_2, *cd*, and *db* in equilibrium. They must then form a closed triangle. Starting with the known force *bc*, we go from B to C. From C, draw CD horizontal. Since this is already drawn, we simply follow the line to D. From D, draw DB to close the triangle BCD. Put on the arrowheads in the direction in which the triangle was drawn, and then place the arrowheads on the same forces at G. This shows *db* in compression, scaled 2,000 lb., and *cd* in tension, scaled 1,730 lb.

A check on the work consists in using joint F, where the forces are now all known, and seeing if these form a closed figure in the force diagram. Start with *ab*. We go from A to B, then from B to D for the force *bd*, then from D to A for force *da*, which brings us back to where we started, the accuracy of the work being thus proved.

108. King Post Trusses.—Let us now consider a king post truss as shown in Fig. 233*a*. We shall assume that the loads carried by the truss have been figured by the methods of the previous articles. Also, assume that the loads on the truss are carried at the joints by purlins and are as shown in the figure. There are no ceiling loads, and the weights of the members of the truss are figured in on the joint loads. Since the load is symmetrical, the reactions are each 4,000 lb. We letter the space diagram as in the previous problems, first taking the loads and reactions, going clockwise, then lettering the spaces within the truss as shown. In constructing the force diagram, we choose a scale to represent a force of 4,000 lb. Since the total load is 8,000 lb., we draw AF twice the unit length. Then AB is 1,000 lb., BC = 2,000 lb., CD = 2,000, DE = 2,000, and EF = 1,000 lb. In drawing the stress lines, no joint can be considered that has more than two unknown forces. A study of Fig. 233*a* will show that joints 1 and 6 are the only joints having no more than two unknown forces. We begin with 1, first using *ga*. We go from G to A. The next force is *ab* as we travel clockwise around

1. We now go from *A* to *B*. Next, draw a line from *B* parallel to *bh*. It is drawn indefinite in length, for *bh* is known in direction only. Since the forces at 1 are in equilibrium, they must form a closed figure in Fig. 233*b*. Then *hg*, being the last force, must close the drawing at *G*, the starting point. Draw a line from *G* parallel to *hg* to meet the line from *B*. This step determines the point *H*. If we put arrowheads at 1, in the direction in which *GABHG* was drawn, *bh* is in compression, for it acts toward 1, and *hg* is in tension, for it pulls away from 1. That is, the lower chord prevents the truss from spreading.

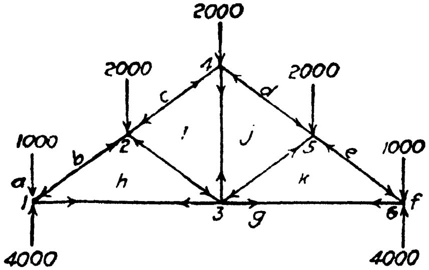


Fig. 233*a*.

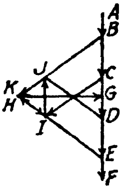


Fig. 233*b*.

We must now go to joint 2, for *ci* and *ih* are the only unknown forces, whereas at 3 there are *hi*, *ij*, *jk*, and *kg*. Begin with *hb*. Go from *H* to *B*, then from *B* to *C*. From *C*, draw a line indefinite in length parallel to *ci*. Now *ih* must close the diagram. Then, from *H*, draw a line parallel to *ih* to meet the line drawn from *C*. The intersection is point *I*. When the arrowheads are put on the forces at 2, *hb* is in compression as found before, *ci* is in compression, and *ih* is also in compression.

We now consider joint 4, where there are but two unknown forces, whereas at 3, there are three. Begin with *ic*, starting at *I* and going to *C*. From *C*, go to *D* to get force *cd*. From *D*, draw a line parallel to *dj*. Since the figure closes at *I*, we draw a vertical line from *I*, to meet the line from *D*. The intersection is the point *J*. The arrowheads put on at 4 show *dj* in compression and *ji* in tension. Next, consider joint 3, beginning with force *gh*. Go from *G* to *H*, from *H* to *I*, from *I* to *J*; then, from *J*, draw *JK* parallel to *jk*. A line from *G* parallel to *gk* intersects the line parallel to *jk*, thus locating *K*, which coincides with *H*. The arrowheads at 3 show *jk* in compression and *kg* in tension.

For a check on the work, consider the forces at joint 5. Begin with *de*. Go from *D* to *E*, then *E* to *K*, *K* to *J*, and *J* to *D*. The figure is closed, it thus being shown that the forces are in equilibrium. The student should now measure all the stress lines in Fig. 233*b*, reduce the results to pounds, and tabulate the value.

Member....	*bh*	*hg*	*ci*	*ih*	*ij*	*jd*	*jk*	*ek*	*kg*
Stress...... Sign........									

109. Truss with Ceiling Loads.—Take a roof truss of the type shown in Fig. 234 carrying a ceiling load in addition to the ordinary dead load. The span l = 42 ft., the pitch = $\frac{1}{3}$, the height being thus 14 ft., and the trusses are spaced 12 ft. apart. The jack rafters are spaced 16 in. from center to center and are 2- by 8-in. timbers. The

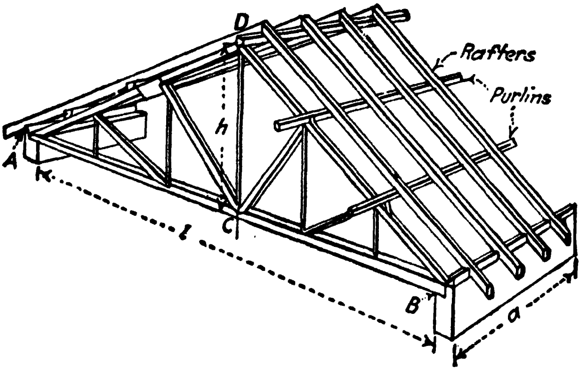


FIG. 234.

purlins are 8 by 10 in. The roof is sheathed with 1-in. spruce sheathing and covered with metal shingles. The ceiling is lath and plaster, held in place by 2- by 6-in. joists, spaced 16 in. center to center.

The first thing is to determine the total weight carried by the truss on the upper chord due to the rafters, purlins, and roof. The length of the upper chord *AD* of Fig. 234 is found as follows: *ACD* is a right triangle. Then

$$\overline{AD}^2 = \overline{21}^2 + \overline{14}^2 = 441 + 196 = 637$$
$$AD = 25.2 \text{ ft.}$$

Now the roof projects beyond the end of the rafters. Suppose we assume that the length of the roof is 26 ft. Since the trusses are spaced 12 ft. apart, the roof area carried by each truss is 26 × 12 × 2 = 624 sq. ft. Handbooks give spruce sheathing 1 in. thick an average weight of 2 lb. per sq. ft. and metal shingles 1 lb. per sq. ft. Then the total roof weight is 624 × 3 = 1,872 lb.

To find the weight of lumber, first find the number of board feet of lumber used. Handbooks give the weight of lumber per cubic foot. Divide this weight by 12, and we have the weight per board foot. This quotient times the number of board feet gives the total weight of the lumber. To find the number of board feet, multiply the breadth and height in inches by the length in feet, and divide the product by 12. Thus for the jack rafters, 2 by 8 in. and 26 ft. long, the number of board feet in one is

$$\frac{2 \times 8 \times 26}{12} = 34.7 \text{ bd. ft.}$$

The rafter would not be 26 ft. long but would be made of two lengths that together made 26 ft. Now, since the rafters are spaced 16 in. apart, the number required from truss to truss is

$$\frac{12 \times 12}{16} = 9$$

on each side. The total number of board feet in the rafters for each truss is

$$2 \times 9 \times 34.7 = 625 \text{ bd. ft.}$$

There are seven purlins 8 by 10 in. and 12 ft. long. Then

$$7 \times \frac{8 \times 10 \times 12}{12} = 560 \text{ bd. ft.}$$

The total number of board feet is

$$625 + 560 = 1{,}185 \text{ bd. ft.}$$

If we use southern pine weighing about 45 lb. per cu. ft., then each board foot will weigh 45 ÷ 12 = 3.75 lb. Call this result 4 lb. The total weight of the lumber is

$$1{,}185 \times 4 = 4{,}740 \text{ lb.}$$

For calculating the weight of a roof truss, textbooks on roofs and bridges give the following formula:

$$W = \frac{AL}{2}\left(1 + \frac{L}{10}\right)$$

Where W = total weight of one truss in pounds,
A = distance between trusses in feet,
L = span of the truss in feet,
A = 12 ft.,
L = 42 ft.,

$$W = \frac{12 \times 42}{2}\left(1 + \frac{42}{10}\right) = \frac{504 \times 5.2}{2} = 1{,}310 \text{ lb.}$$

The total load carried at the joints of the upper chord is

	Load, Pounds
Roof covering	1,872
Rafters and purlins	4,740
Truss	1,310
Total	7,922

Textbooks on graphic statics give for a roof with covering such as we assumed a weight of 10 to 12 lb. per sq. ft. of roof surface to care for covering rafters and purlins. In our problem the roof area is 624 sq. ft. If we assume 11 lb. per sq. ft., then the total weight is

$$11 \times 624 = 6{,}864 \text{ lb.}$$

Our calculation, 1,872 + 4,740 = 6,612, agrees quite closely with this result.

Assume a snow load of 25 lb. per sq. ft. of horizontal projection of roof or, what amounts to the same thing, the floor covered by the roof. This area is 42 × 12 = 504. Then snow load is

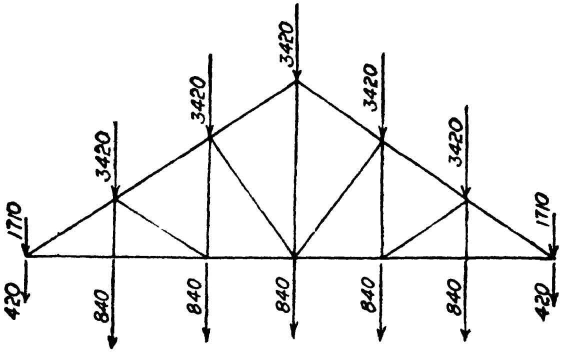


Fig. 235.

$$504 \times 25 = 12{,}600 \text{ lb.}$$

The entire upper chord load is

$$7{,}922 + 12{,}600 = 20{,}522; \text{ say } 20{,}500 \text{ lb.}$$

Figure 235 shows that there are five joints of the truss carrying full loads and two end joints carrying half loads. The load is then divided into six equal parts, or 20,500 ÷ 6 = 3,420 lb. for each joint except the ends, which carry 1,710 lb. each. Figure 235 shows the truss and loads.

In figuring the ceiling loads, we may calculate them by the method previously used, taking plastering and lath from 6 to 8 lb. or using 10 lb. per sq. ft. to include the plastering, lath, and joists. Since there are 42 × 12 = 504 sq. ft. of ceiling, the total ceiling load is 10 × 504 = 5,040 lb. There are six panel loads on the lower chord. Then

$$5{,}040 \div 6 = 840 \text{ lb.}$$

is the joint load for the lower chord except the ends; the load for these is 840 ÷ 2 = 420 each. Figure 235 shows the load. The total load carried by the truss is

$$20{,}500 + 5{,}040 = 25{,}540$$

Since the load is symmetrical, each reaction is one-half the load, or 12,770 lb. Now, instead of having 1,710 lb. and 420 lb. each at *A*, with 12,770 lb. up, it is customary to

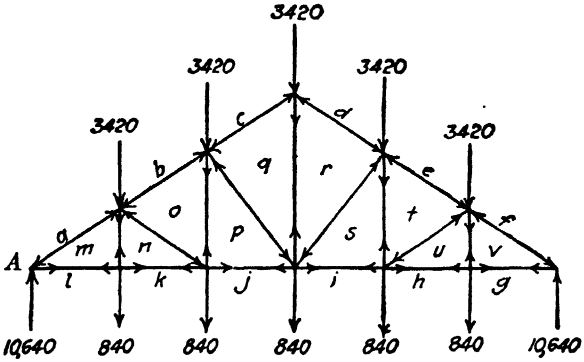


Fig. 236.

replace them by one load, the resultant. This force is 12,770 − 1,710 − 420 = 10,640 lb. This change of forces has no effect on the resultant stresses and makes the work easier because there are two less forces to work with. Figure 236 shows the loads and lettering. Represent the spaces by small letters, beginning with the space to the extreme left and lettering the loads and reactions clockwise. Then, take the spaces formed by the members of the truss. going from left to right. For the force diagram (Fig. 237), choose a line ½ in. long to represent 3,420 lb. Since there are five such loads, we draw a line 2.5 in. long to represent the loads on the upper chord and thus determine the points

A, *B*, *C*, *D*, *E*, and *F*. The next load is the right reaction of 10,640 lb. We then go from *F* to *G*, to scale, then 840 down to *H*, 840 to *I*, etc., until *L* is located. The next load is the left reaction upward, which takes us from *L* to *A*. The accuracy of the work up to this point, now determined as *LA*, should scale 10,640 lb.

We now begin with the joint at the left end because it has but two unknown quantities, starting with force *la* because it is completely known. From *L*, go to *A*. From *A*, draw a line parallel to *am*, indefinite in length, to meet a line drawn through *L*, parallel to *lm*. This locates point *M*. Placing arrowheads at the joint, in the same direction taken in drawing *LAML*, shows *am* in compression and *ml* in tension.

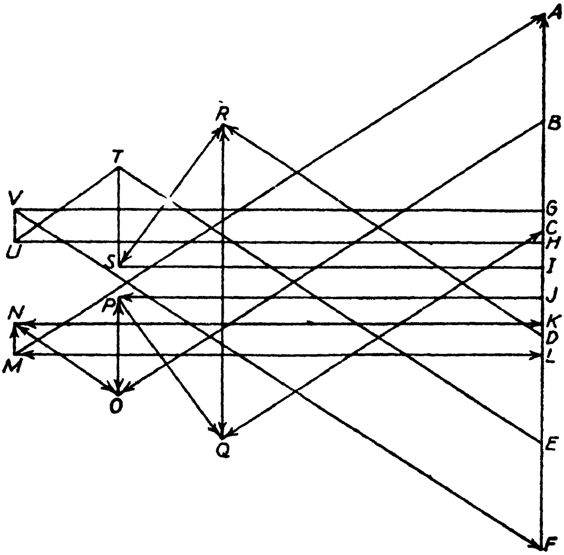


FIG. 237.

Now, go to the joint on the lower chord formed by *mn*, *nk*, *kl*, and *lm*. Go from *K* to *L* and from *L* to *M*. From *M*, draw a parallel to *mn*, to meet a parallel to *nk* through point *K*. This locates point *N*. Stresses *mn* and *nk* are found to be tension.

Next, use the second joint in the upper chord, starting at *N*. From *N* go to *M*, then from *M* to *A* and from *A* to *B*. Now, from *B*, draw a line parallel to *bo*, indefinite in length. The diagram must close at *N*. From *N*, draw a line parallel to *on* to meet the line parallel to *bo*. This step determines the point *O*. Inserting the arrowheads as before shows *nm* in tension, *ma* compression, *bo* compression, and *on* compression.

Take next the third joint of the lower chord, beginning at *J* because force *jk* is completely known. From *J*, go to *K*, from *K* to *N*, from *N* to *O*. Now, from *O*, draw a line parallel to *op*; from *J*, draw one parallel to *pj*. The intersection locates point *P*. The arrowheads show *kn* in tension, *no* compression, *op* tension, and *pj* tension.

Now, take the third joint of the upper chord, starting with point *P*. From *P*, go to *O*, from *O* to *B*, from *B* to *C*. From *C*, draw a line parallel to *cq* and from *P* a line parallel to *qp*. The intersection gives joint *Q*. *po* is in tension, *ob* compression, *cq* compression, and *qp* compression. The reader has undoubtedly noticed that the stress in any member is found at two different joints. The results should give the same kind of stress at both joints.

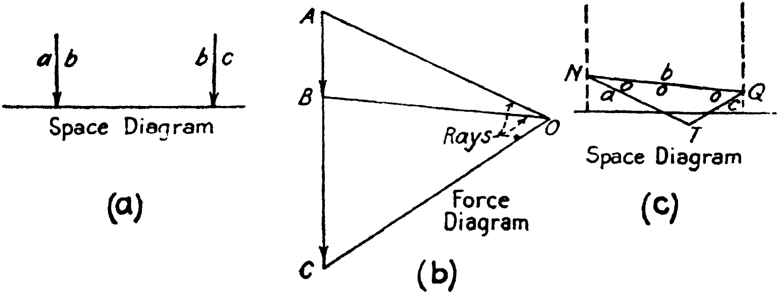
Next, use the joint at the vertex of the roof. From *Q*, go to *C* and from *C* to *D*. Through *D*, draw a line parallel to *dr*, to meet a parallel to *rq*, through the point *Q*. This locates *R*. The diagram shows *dr* in compression and *rq* in tension.

Because of symmetry, stress *mn* = *uv*, *am* = *fv*, etc. However, as a check on the drawing, the student should continue the operation through all the joints.

110. The Equilibrium Polygon.—The principle of moments is generally used to determine the reactions for a beam or truss. However, there is a graphical method, known as the *equilibrium polygon method*, that may be easily applied in determining the position of the resultant of nonconcurrent forces or in finding the reactions.

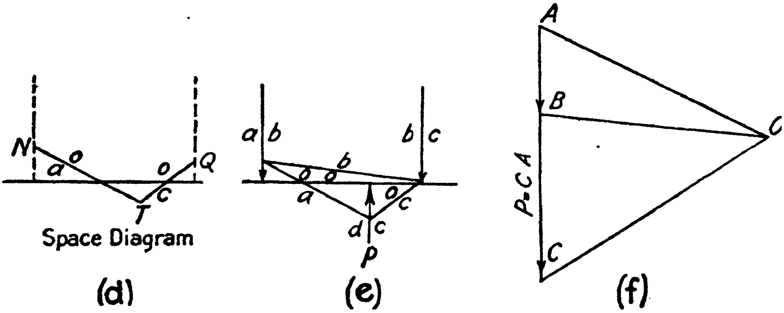
The method will first be applied in finding the resultant of two parallel forces. Let *ab* and *bc* represent the lines of action of the forces as in Fig. 238*a*, which is called the *space diagram.* The forces are then laid off to scale and in order from left to right, as in Fig. 238*b*, which is known as the *force diagram.* The resultant of the forces is their vector sum, or AC, and it now remains to determine its position.

In the force diagram, select any point O, and draw the rays OA, OB, and OC. These rays are the components of the forces AB and BC. The procedure is essentially this: Since the forces cannot be produced to an intersection, they are resolved into components, whose lines of action will intersect each other.



Figs. 238*a-c*.

In the force diagram, the point O may be selected anywhere, for any components AO and OB have as their resultant the force AB and are therefore equivalent to the original. Since a force may be assumed to act at any point on its line of action, some point as N on *ab* is selected, and the force AB is resolved into the two components AO and OB, whose lines of action *oa* and *ob* will be parallel to these components. These lines of action are called ***strings.*** From an examination of the force diagram, it is seen that OB is also a component of BC, and the string *ob* is therefore drawn from point N to string *bc*. This latter point may be called *Q*. At *Q* the force *BC* is resolved into the two components *BO* and *OC*. In place of the original forces *AB* and *BC*, whose lines of action are *ab* and *bc*, equivalent components have been substituted, acting along the strings *oa*, *ob*, and *oc*. Figure 238*c* shows the lines of action of the substitute forces. But there are equal and opposite components acting along string *ob*. In the force diagram, it can be seen that the component *OB* of the force *AB* acts in the direction from *O* to *B*, whereas the component *BO* of the force *BC* acts from *B* to *O*. Since these two components neutralize each other, Fig. 238*c* may be simplified by removing the string *ob*, and the result is Fig. 238*d*.



FIGS. 238*d-f*.

By the foregoing substitutions, the original forces *AB* and *BC* have been replaced by the components *AO* and *OC*, whose lines of action are *ao* and *oc*. Since these strings intersect at the point *T*, this will be a point on the line of action of the resultant. The direction of the resultant is determined from the force diagram, and with point *T* known the line of action of the resultant can be drawn.

If, in addition to the two forces, another force *P* equal and opposite to the resultant and acting through point *T* is added to the system, then the forces will be in equilibrium. Since the components of the force *P* will be *CO* and *OA* and the lines of action *oc* and *oa*, it may be seen that there are equal and opposite forces along the strings. It should therefore be evident that when a system of forces is in equilibrium the forces are resolvable into equal and opposite components along each of the strings.

111. Reactions.—The foregoing method will now be applied in finding the reactions for the beam in Fig. 239*a*.

As before, draw the force polygon by laying off *AB* and *BC* to scale (Fig. 239*b*). Since reaction must equal action, then $R_2 + R_1$ must be equal and opposite to *AC*. The problem now consists in finding a point *X* that will divide *CA* into the two parts representing R_2 and R_1. That is, some segment as *CX* is equal to R_2, and *XA* is equal to R_1. Select any point *O*, and draw the rays *OA*, *OB*, and *OC* in the force diagram. Then, in Fig. 239*a* at any point on *ab*, resolve the force *AB* into its components whose lines of action will be parallel to the corresponding rays in the force polygon. The string *ob* is drawn between *ab* and *bc*, for it represents the line of action of a component that is common to forces *AB* and *BC*. The strings *oa* and *oc* are parallel to components *OA* and *OC*. But *OA* is a component of *AB* and also of *XA* or R_1; therefore, the string *oa* is drawn from the line of action *ab* to the line of action of R_1, and *oc* is drawn from the line of action *bc* to that of R_2.

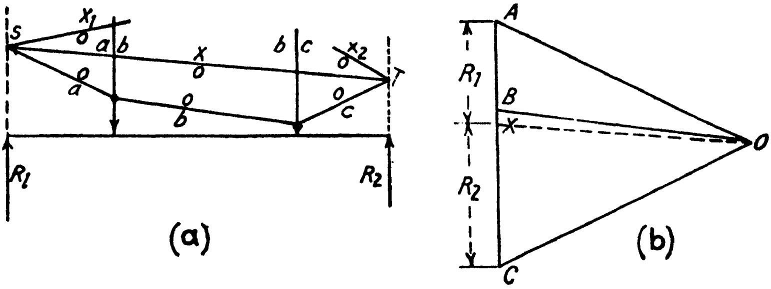


FIG. 239.

At point *S* on R_1, resolve R_1 into a component *OA* whose line of action is *oa*, and some other component, whose line of action is ox_1. At point *T*, resolve R_2 into a component *CO*, whose line of action is *co*, and some other component, whose line of action is ox_2. Thus far, there are component forces of equal and opposite values acting along each of the strings oa, ob, and oc. For the forces along the string ox_1 and ox_2 to neutralize each other, these strings must coincide, and they must therefore have the direction ST. The force polygon, also, should show components corresponding to this string. The space diagram makes it possible to determine the direction ox of the components OX and XO; but the amount of these components will be found by drawing a ray through O parallel to ox, and where this intersects CA will be the point X. This point X divides CA into two parts, CX or R_2 and XA or R_1.

At point S there are two components XO and OA; and at point T, CO and OX. But XO is equal and opposite to OX, and so these forces neutralize each other. Since component forces that neutralize each other have been found for the other strings, the forces as a whole are in equilibrium. In determining R_1 and R_2, the string polygon must close in order to make equilibrium possible, and therefore the direction of the string ox is known. But ox is the line of action of a component XO that is common to both R_2 and R_1. To determine this component, a line is drawn through O in the force diagram, parallel to ox. This ray OX represents the required component and, by determining the point X, divides CA into the two reactions R_2 and R_1. The method may be extended with little difficulty to include nonparallel forces, for these forces are in any event replaced with concurrent components.

112. Fixed Ends.—When both ends of a truss are fixed, the reactions for a wind load cannot be determined unless some assumption is made. If the roof is comparatively flat, the normal components of the wind load are nearly vertical. If the assumption is made that the reactions are in the same direction as the normal component of the wind, the results will be close enough for practical purposes. However, if the slope is large, the assumption is made that the horizontal components of the reactions are equal. It is understood that both supports are equally elastic and both ends of the truss fastened in the same way.

When a roller is under one end of a truss, the reaction at that end is vertical, if we assume that the friction under the roller is very small. The fixed end then takes all the horizontal pressure of the wind.

We shall give three illustrations showing how the reactions are computed graphically for the different end conditions.

Figure 240 is the drawing of a roof truss with the total normal component of the wind load acting at the truss joints. The assumption is made in this case that the reactions are parallel and in the direction of the loads.

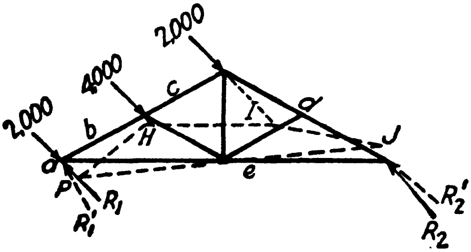


Fig. 240.

They are represented by R_1 and R_2. The diagram is lettered as in previous cases. We now draw the force diagram *ABCD* parallel to the joint loads. Choose any point *O* as a pole and draw *OA*, *OB*, *OC*, and *OD* as in Fig. 241. Next, choose any point *P* on the reaction R_1, and draw a line parallel to *OA* to met the force *ab*. But *ab* and R_1 coincide in directions, so that the point *P* is on both forces and no line is really drawn. Now, from *P*, draw a line parallel to *OB* to meet *BC* at *H*. From *H*, draw a line parallel to *OC* to meet *cd* at *I*, and then, from *I*, a line parallel to *OD* to meet R_2, the right reaction, at *J*. Join *P* and *J*. From *O*, now draw a line parallel to *PJ* to meet *AD* at *F*. Then R_2 is equal to *DF*, and *FA* is equal to R_1. These lines may now be scaled and reduced to pounds from the unit *AB* = 1,000 lb.

We shall now determine the reactions, making the assumption that their horizontal components are equal. Authorities seem to agree that, for fixed ends, this condition is the one most nearly realized. Use the same truss and loads as in Fig. 240. The vertical components of the reactions are independent of the end conditions of the truss, depending only on the load. First assume the reactions parallel, and find them as in Fig. 241.

Now the vertical and horizontal components of a force may be found by drawing a vertical line through one end of the line and a horizontal through the other end, forming a right triangle. The two legs of this triangle are the two components of the force. Thus, in Fig. 242, the components of *FA*, the left reaction, are *FH* and *HA*. Also, the components of *DF* are *DG* and *GF*. Assuming that the reactions are parallel, we see from the drawing that the horizontal components *GF* and *FH* are not equal. If the components are assumed equal, we find the center point of *HG* at *S*. Join *SA* and *SD*. *SA* is the left reaction in magnitude and direction and *DS* is the right. That this condition is true is evident, for their sum forms a closed figure with the sum of the loads *AD*. The reactions drawn in dotted lines are represented by R_1^1 and R_2^1 in Fig. 240. Their magnitude may now be found by measurement.

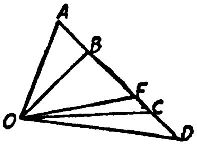


FIG. 241.

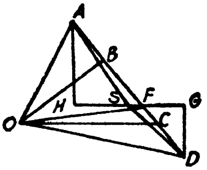


FIG. 242.

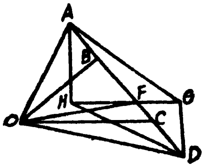


FIG. 243.

When there is a roller under one end of a roof truss, the reaction at that end is vertical for all practical purposes. As has been said before, the vertical components of the reactions are independent of the end conditions of the truss. Then proceed as with reference to Figs. 240 and 241, assuming that the reactions are parallel and finding *DF* and *FA* as before. Now, if the roller is under the right end of the truss, the reaction R_2 is vertical. Then, through D and A (Fig. 243), the ends of the force diagram, draw vertical lines. Through F, draw a horizontal line to meet the vertical lines through A and D at H and G. Then DG is the reaction R_2. But HA is the vertical component of the left reaction R_1 and GH its horizontal component, for R_1 is to carry the entire horizontal-component wind pressure. Then join GA, and we have the left reaction R_1 in magnitude and direction.

If the roller is under the left end of the truss, this reaction is vertical and equal to HA of Fig. 243. The right-hand reaction R_2 takes the entire horizontal component GH and has a vertical component DG. The right reaction is then found by joining DH. It is now known in magnitude and direction. Figure 244 shows the truss and its load. R_1 and R_2 are the reactions for the case when the roller is under the right, or leeward, side of the truss, and $R_1{}^1$ and $R_2{}^1$ are the reactions when the roller is under the left, or windward, side.

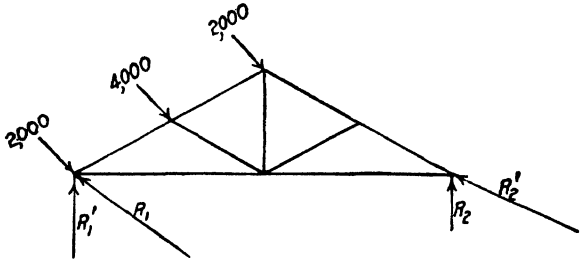


Fig. 244.

The methods shown, though applied to a special truss, are applicable to any similar case and if applied in this way will give good results. The student should be careful to choose the arbitrary points designated by O and P so that the drawing comes entirely on the paper and acute intersections are eliminated if possible.

113. Stresses Due to Wind Load. *Graphical Methods of Computation.*—Figure 245 shows a roof truss of angle

30°; the distance between the joints of the upper chord is 10 ft. The trusses are spaced 12 ft. apart. There is a roller under the left end.

From the table of Art. 104, the normal component of the wind for a 30° inclination is 27 lb. For easier calculation, 30 lb. per square foot of roof surface is assumed. The load carried is 10 × 12 × 30 = 3,600 lb. at each full joint and 1,800 lb. at the half joints. Figure 245 shows the loads and lettering for the space diagram.

As in the previous article, the assumption is first made that the reactions are parallel to the wind loads, and they are designated in this position by $R_1{}^1$ and $R_2{}^1$. The force diagram (Fig. 246), *ABCDE-O* is drawn. Beginning at *t* of Fig. 245 on $R_1{}^1$ and *ab*, we draw a line parallel to *OB* to meet *bc* at *u*, etc., thus locating *t*, *u*, *v*, *w*, *x*. The line *tx* determines the direction in which to draw a line from *O* to cut *AE* in *I′*. Now *EI′* and *I′A* would be the reactions under the first assumption. But there is a roller under the left end, and therefore the reaction is entirely vertical. All the horizontal component of the wind is taken by the right support. Then, through *A*, draw a vertical line to meet a horizontal line through *I′* at *I*. Then *EI* is the right reaction R_2, and *IA* the left reaction R_1.

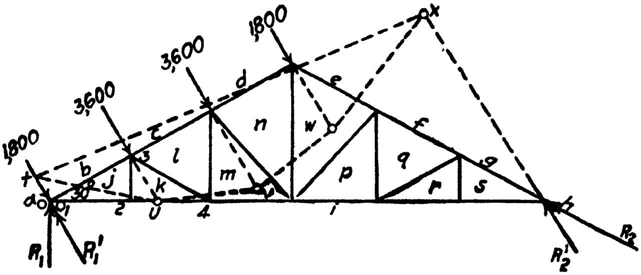


FIG. 245.

Since there is no wind load on the right side of the roof, there are no loads *ef*, *fg*, and *gh*. This fact is shown in the space diagram of Fig. 246, where *E*, *F*, *G*, and *H* coincide; in any stress diagram, if two letters are found to fall at the same point, the stress is zero in the member between those letters.

To construct the stress diagram, begin at the left-hand joint, starting with force R_1; go from I to A, then A to B. From B, draw a line parallel to bj to meet a horizontal line through I, parallel to ji. This step determines J. All the stresses at joint 1 are now known. Next, take joint 2, beginning with IJ. Now, draw through J a parallel to ki. But this makes J and K coincide, indicating no stress in jk.

At joint 3, begin with K, go to J and from J to B, B to C; then, through C, draw a parallel to cl, to meet a line through K, parallel to lk. This step determines the joint L.

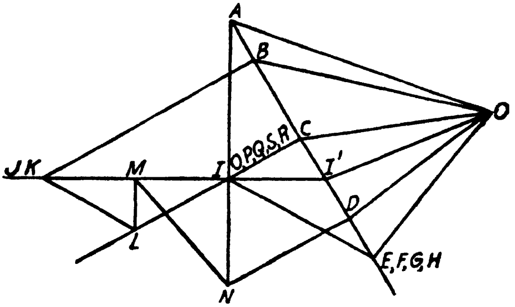


Fig. 246.

In this way, we proceed from joint to joint until all the joints are located. The diagram shows that I, O, P, Q, S, and R coincide; thus, there are no stresses in pi, ri, si, pq, op, qr, and rs.

Exercise 1.—Assume the wind blowing against the right side of the truss. Find the reactions and stresses in all the members of Fig. 245.

Exercise 2.—If 40 lb. per sq. ft. is assumed for the dead load, construct the stress diagram for Fig. 245.

Exercise 3.—Scale the three diagrams of Fig. 245. Combine the stresses due to the dead load with the stresses from Fig. 246 in the corresponding members and also with the stresses from example 1. In this way, find the maximum stress in each of the members.

114. Stresses in a Fink Roof Truss.—In all roof trusses that have been analyzed thus far, no difficulties have been encountered in finding the stresses in the members because we never found a case in which there were more than two unknowns at any joint. However, there is a common type of steel truss, of large span, in which one joint is found at which there are three unknowns. The analysis of this type of truss requires the introduction of an extra line called an "auxiliary line."

Figure 247 shows a Fink truss. We shall assume that the full joint loads are 1,200 lb. The methods for determining these loads have been sufficiently discussed and need not be repeated here. The left end of the truss is assumed fixed and the right end assumed to be on rollers.

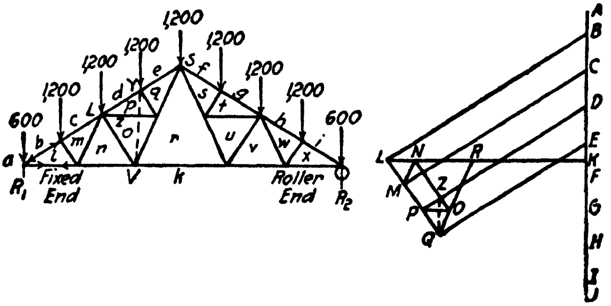


FIG. 247.

The straight line *AJ* is drawn vertically for the force diagram as shown. *AB* to scale equals 600 lb. Since the truss is symmetrically loaded, the reactions at the ends are equal, and each is ½*JA*. This step locates the point *K* at the center of the line. We now begin with the extreme left joint.

Go from *K* to *A* and from *A* to *B*; draw a line through *B*, parallel to *bl*, to meet a line through *K*, parallel to *lk*. This step locates point *L* and shows *bl* in compression and *li* in tension. Now, take the second joint in the top chord. Go from *L* to *B* and from *B* to *C*; through *C*, draw a line parallel to *cm* to meet a line through *L*, parallel to *lm*. This procedure locates *M*. Next, take the second joint on the lower chord. Beginning at *K* and going around the joint clockwise, we locate joint *N*.

When we attempt to analyze the third joint of either the top or the bottom chord, we find three unknowns. Now, imagine the members *po* and *pq* removed from the structure and a member *VY* put into the truss. The structure will be rigid, for we have two triangular frames *LVY* and *YVS*. The line *YV* is called the auxiliary line and is drawn dotted to show that it is not a part of the truss. The introduction of *YV* does not affect the stresses in the other members of the truss.

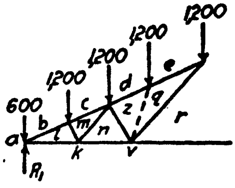


FIG. 248.

In order that the student may not be confused, Fig. 248 is drawn showing one side of the roof truss without *op* and *pq* but with the auxiliary line.

We now go to the third joint of the top chord, beginning with *N*. Go from *N* to *M*, *M* to *C*, *C* to *D*. From *D*, draw a line parallel to *dz* to meet a line through *N*, parallel to *zn*. This determines the point *Z* of Fig. 247.

Now, consider the fourth joint of the top chord. Go from *Z* to *D* and *D* to *E*. From *E*, draw a line parallel to *eq* to meet a line from *Z*, parallel to *zq*. This locates the point *Q*. Since the point *Q* is located, we may now refer back to joint 4 of the top chord in Fig. 247. Beginning at *D*, go to *E* and from *E* to *Q*. From *Q*, draw a line parallel to *pq* to meet a line from *D*, parallel to *dp*. This step locates *P* and gives the stress in *pd* and *pq*. The remaining stresses may now be found by completing the stress diagrams as shown. The direction of these stresses should be found and the magnitudes scaled and tabulated.

If the snow load is not included in the 1,200-lb. joint loads, the stresses caused by snow may easily be found from the stress diagram of Fig. 247. For example, if the snow causes a full joint load of 600 lb., then the stresses will be just one-half of those in Fig. 247.

If the snow joint loads were 900 lb. then the stresses would be three-quarters of the dead-load stresses.

When the wind load is considered, two cases arise, *viz.*, wind on the left side of the truss and wind on the right.

Figure 249 shows the truss with the wind blowing against the left side. No definite amounts are assumed. The length of the vectors at the joints could be used for any pressure per square foot.

As in previous articles, the asumption is first made that the reactions are equal and parallel to the loads. The force polygon *A*-*J* is drawn, and the points are joined to *O*. Next the funicular polygon 1, 2, 3, 4, 5, 6 is drawn in dotted lines. The direction 1–6 drawn from *O* determines *K′*. Through *K′*, draw a horizontal line to meet the vertical through *F*. This procedure locates *K*. Then *FK* is $R_2{}^1$, and *KA* is R_1. This condition agrees with the assumption that the roller is under the right end and the reaction $R_2{}^1$ entirely vertical.

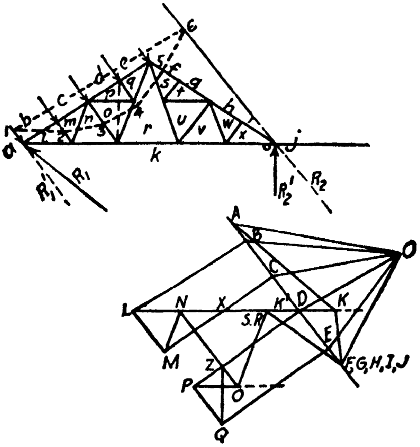


FIG. 249.

We now begin at the left-half joint, locating *L*. Next, the second joint on the top chord, locating point *M*, and then the second joint on the lower chord, locating *N*. We are unable to proceed from this point because of three unknowns at every joint. Then, as in Fig. 247, we assume that *op* and *pq* are removed and an auxiliary line *YV* is drawn dotted. We are now able to proceed with the third joint of the top chord. Go from N to M, M to C, C to D. Then, from D, draw a line parallel to dz to meet a line from N, parallel to nz. This step locates point Z. Now, begin with joint 4 of the top chord. Go from Z to D, and from D to E. Then, from E, draw a parallel to eq to meet a line from Z, parallel to zq. This construction locates point Q.

The auxiliary line may now be removed and op and pq replaced. From the fourth joint of the top chord, we go from D to E and from E to Q. Then a parallel to pq is drawn from Q to meet a parallel to dp from D. This step locates point P. We now finish with the third joint of the upper chord for locating O, then with the third joint of the lower chord for locating R, etc., until all the remaining points are located. It is found that points R, S, T, U, X coincide, this fact indicating zero stress in rs, st, tu, uv, vw, wx. The student may now trace out the stress diagrams for tension and compression members, scale the lines, and find the numerical values of the stresses. These results should be tabulated and the total stress in the members should be found by combining dead, snow, and wind-left diagrams.

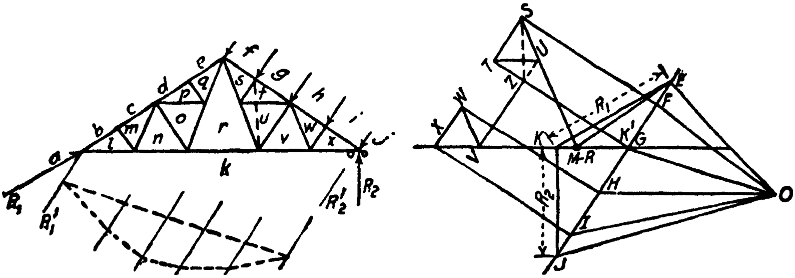


Fig. 250.

The truss with wind on the right is shown in Fig. 250 with the drawings for the determination of reactions and stresses. No discussion will be given for this case. It is left to the student to trace every step, tabulate the stress values, and combine with the wind and snow stresses.

Problems

200. The members in one plane of a four-cornered tank tower are shown in Fig. 251. The weights given are due to the tank and water above. The horizontal forces are due to the wind. What sizes of steel rods may be used as tension members?

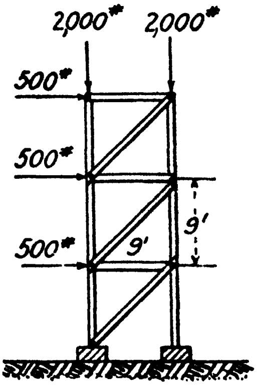


FIG. 251.

201. The Howe truss shown in Fig. 252 rests on rollers at one end and is fixed at the other end. The trusses are spaced 16 ft. apart. The weight of each truss may be taken as 2.5 lb. per sq. ft. of horizontal area, the roof covering as 10 lb. per sq. ft., and the snow load as 20 lb. per sq. ft. of roof. The wind load may be taken as the equivalent of a vertical load of 22.5 lb. per sq. ft. of roof surface. Analyze the truss for dead- and snow-load stresses.

202. The Pratt truss (Fig. 253) used for the railroad bridge has six equal panels. The dead-load weight of the truss per linear foot may be computed by the formula $W = 5l + 350$, where l is the span in feet. The weight of the track may be taken as 400 lb. per lin. ft. and the live load as 3,500 lb. per lin. ft. Assume the weight of truss to be supported equally at the lower-chord points. What are the stresses?

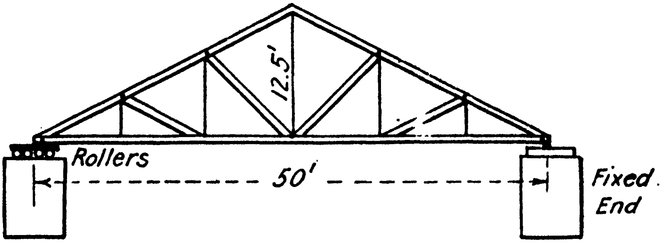


FIG. 252.

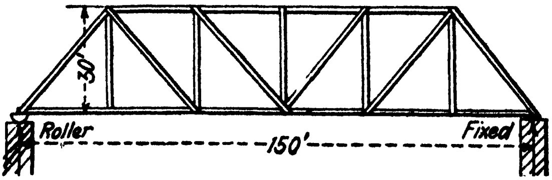


FIG. 253.

203. The saw-tooth truss (Fig. 254) is much used for factory buildings. The truss is unsymmetrical, and the steep surface containing the skylight is toward the north. In the problem, the trusses are 25 ft. apart. The rise is 6.72 ft. On account of the flatness of the roof the combined action of the wind and snow loads may be assumed as equivalent to a vertical load of 25 lb. per sq. ft. of roof surface. The weight of the truss may be taken as 1.5 lb. per sq. ft. of roof and the weight of the roof covering as 7 lb. per sq. ft. of roof. What are the stresses?

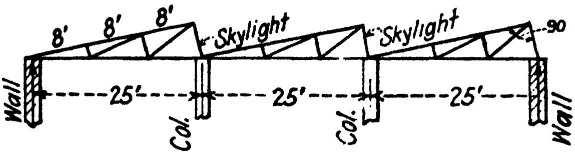


Fig. 254.

204. In Fig. 255, the distance between trusses b is 18 ft. The weight of the trusses may be computed approximately by the formula $W = bl^2/24$.

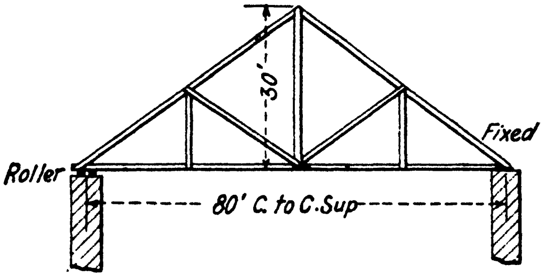


Fig. 255.

The wind load is taken as 40 lb. per sq. ft. of vertical surface and the snow load as 30 lb. per sq. ft. of horizontal surface. Assume that the weight of the truss is supported equally at the upper chord points. What are the stresses?

205. A steel head frame for hoisting ore from a mine is shown in Fig. 256. The frame is pinned at A and anchored at B, so that the reaction at B may act either upward or downward. The length of AC is 90 ft. and of AB is 20 ft. The inclined part of the hoisting rope makes an angle of 50° with the horizontal. What are the amount and kind of stress in each member?

206. In the Fan truss of Fig. 257, the weight of roof and truss may be taken as 12 lb. per sq. ft. of surface and applied at the panel points of the upper chords. The ceiling load may be taken as 12 lb. per sq. ft. and applied at the panel points of the lower chord. The snow

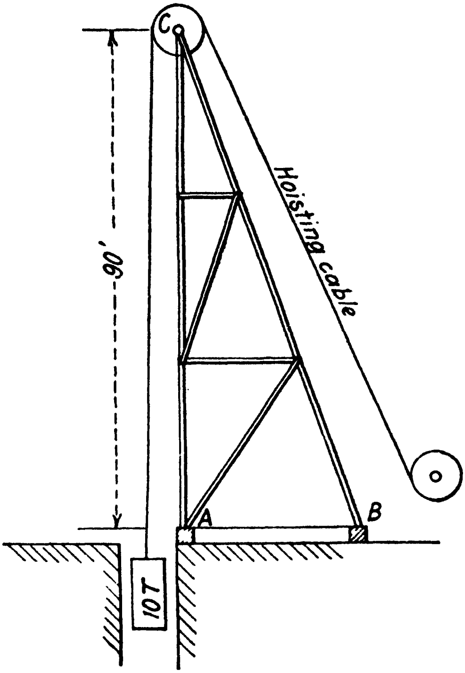


FIG. 256.

and wind loads are combined into a single vertical force of 25 lb. per sq. ft. of surface. What are the stresses due to the combined loads? Trusses are spaced 16 ft. apart.

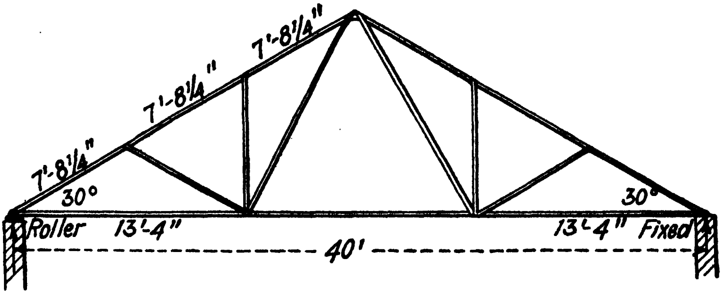
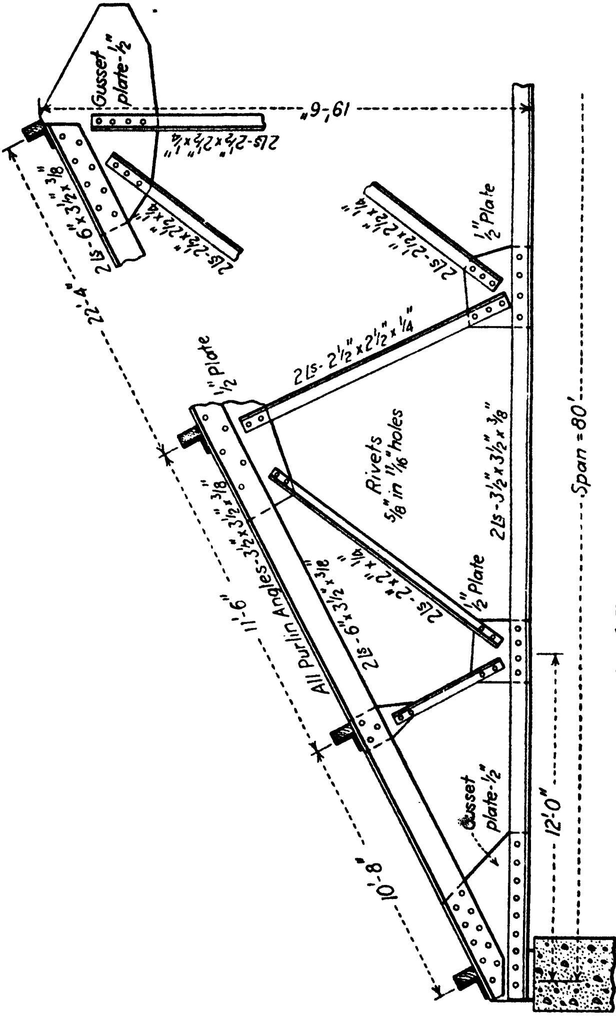


FIG. 257.



Roof Truss showing some Details

FIG. 258.

CHAPTER X

STABILITY AND CENTER OF GRAVITY

CENTER OF GRAVITY

115. Discussion.—Every particle or part of a body possesses weight. *Weight* is the force with which the body is pulled toward the center of the earth and is proportional to the mass of the body of which it is the weight. All weights are parallel forces, practically speaking, being directed toward the center of the earth. By Art. 77, they may be combined into a resultant equal to their algebraic sum.

The point at which an equal and opposite force may be applied to produce perfect balance, or equilibrium, is called the *center of gravity.*

Or *if the moment of the weight of each part of the body is taken about any axis and the moments are added, then the center of gravity is the point where the resultant weight may be applied so that its moment about the same axis is equal to the sum of the moments of all the weights, or gravity forces.*

It is beyond the scope of this work to show the application of this definition in determining the center of gravity, since the body must be divided into a very large number of parts before the entire body is included. The summation of the moment of the forces requires integral calculus. Fortunately, the principal application of the theory of the center of gravity in engineering mechanics is with reference to the plane area made up of rectangles, triangles, and circles. and the problem is not a difficult one.

116. Center of Gravity by Experiment and Balancing.—The center of gravity of some objects may be found by balancing the object on a point. Take a thin plate of thickness *t*, shown in Fig. 259. Draw the diagonals of the lower and upper faces to intersect at *P* and *I*, respectively. If the plate is placed on a vertical stick with a small end or point, with the point at *I*, the plate will not rotate. That is, it is balanced. If suspended from *P*, the plate will hang horizontally. The center of gravity of the plate is at the center of the line *PI*.

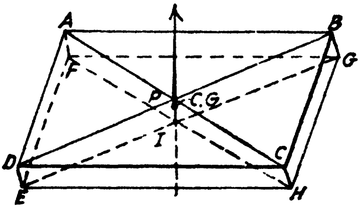


FIG. 259.

The center of gravity of the area of the cross section of irregular bodies such as a rail shown in Fig. 260 may be found by a simple experiment. Cut a piece of cardboard to the shape of the cross section, and find the balancing point. This will give the position of the center of gravity for all practical purposes.

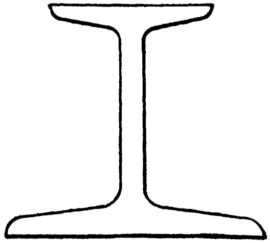


FIG. 260.

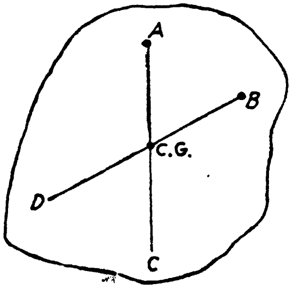


FIG. 261.

Another method for determining the center of gravity is by suspension. Take an object, the section of which is shown in Fig. 261. Suspend it from some point *A*. The body will not come to rest until its resultant weight is vertically downward from *A*. Therefore, through *A*, draw a vertical line *AC*. Then suspend the body from a point *B*, and let it come to rest. Through B, draw a vertical line BD. The point of intersection of AC and BD is the position of the center of gravity.

The student should try each of these methods to familiarize himself with them.

117. Center of Gravity of an Area.—Strictly speaking, there is no center of gravity of an area, for an area does not have weight. The point I, at the intersection of the diagonals of the face $ABCD$ of Fig. 259, is the center of area of a rectangle. Because of its relation to the center of gravity of the plate, it is spoken of as the "center of gravity of the rectangle."

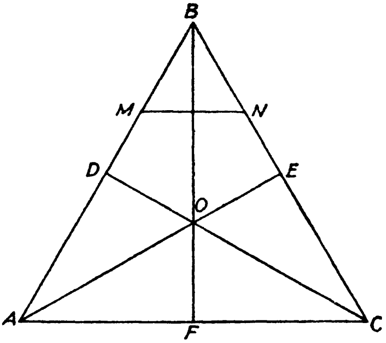


FIG. 262.

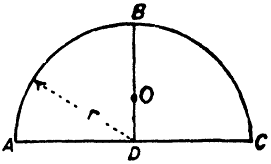


FIG. 263.

The center of gravity of a line or rod, uniform in section, is at its center.

If a body or an area has an axis or line of symmetry, the center of gravity is somewhere on that line.

The median line BF of the triangle ABC of Fig. 262 bisects any line MN parallel to the base AC. Then the center of gravity of the triangle is somewhere on BF. It is also on the median AE. Then the center of gravity of the triangle is at the intersection O of the medians. But $BO = \frac{2}{3}BF$. If h is the altitude of the triangle, *the center of gravity is* $\frac{2}{3}h$ *distant from the vertex or* $\frac{1}{3}h$ *from the base.*

The center of gravity of a *semicircle* ABD (Fig. 263) is on the bisecting radius DB at the point O, and

$$DO = \frac{4r}{3\pi} \quad \text{or} \quad y = \frac{4r}{3\pi} \qquad (32)$$

The center of gravity of a *circle* is at its center.

If a body has *two axes of symmetry*, the center of gravity is at the intersection of these axes. Take an I beam, for example, shown in Fig. 264. *AB* and *CD* are axes of symmetry. Then the center of gravity of the cross section is at *O*, the intersection of the axes.

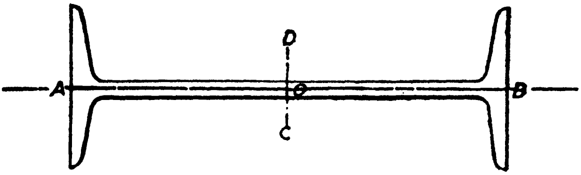


FIG. 264.

118. Practical Applications.—Many problems in engineering practice require the center of gravity of the cross-sectional areas of angle irons, T beams, channels, etc. Most sections do not have two axes of symmetry, and the center of gravity must be found by calculations. Fortunately, in most cases, the area may be divided into rectangles, triangles, circles, or combinations of these figures. The determination of the center of gravity is then not difficult.

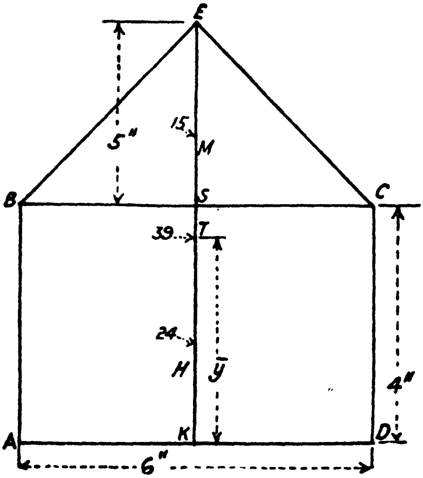


FIG. 265.

The method for the calculation of the center of gravity will be developed by solving some examples.

Example 1.—Find the center of gravity of the area shown in Fig. 265.

Solution.—Since EK is an axis of symmetry, the center of gravity, is on this line. Choose AD as the axis of reference, and represent the distance from AD to the center of gravity by $\bar{y}$.

Imagine a force of as many pounds as there are square inches in the area, applied at H, the center of gravity of the rectangle, and perpendicular to the plane of the area. This will be a force of 4 × 6 = 24 lb. with moment arm $KH = 2$. At M, the center of the triangle, apply a force of $(6 \times 5/2) = 15$ lb. Now, $SM = \frac{1}{3} \times 5 = \frac{5}{3}$. Then KM, the moment arm of 15 lb., $= KS + SM = 5\frac{2}{3}$ ft. But the center of gravity of any area or body is the point where the entire area may be concentrated so that its moment about an axis is equal to the sum of the moments of the separate areas, weights or forces, about the same axis.

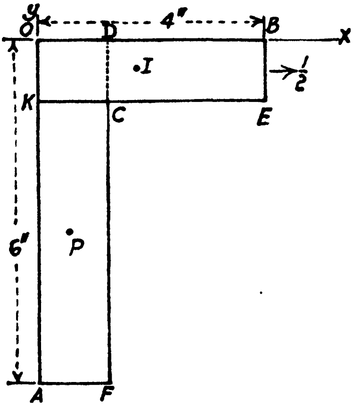


FIG. 266.

Let T be the point where the total area of 15 + 24 = 39 is concentrated, with a moment arm $KT = \bar{y}$. Then

$$39 \times \bar{y} = 24 \times 2 + 15 \times 5\tfrac{2}{3}$$
$$\bar{y} = \tfrac{133}{39} = 3.41 \text{ in.}$$

If the body has no axes of symmetry, it is necessary to obtain the position of the center of gravity with reference to two axes called the x and y axes. The coordinates of the center of gravity are represented by $\bar{x}$ and $\bar{y}$. The formulas for obtaining these coordinates are as follows:

$$\bar{x} = \frac{\text{sum of the moments of the areas about the } y \text{ axis}}{\text{sum of the areas}}$$
$$= \frac{\mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2\mathbf{x}_2 + \mathbf{A}_3\mathbf{x}_3 + \cdots}{\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \cdots} \qquad (33)$$

where x_1, x_2, x_3, etc., are the distances from the y axis to the center of the areas A_1, A_2, A_3, etc., respectively.

$$\bar{y} = \frac{\text{sum of the moments of the areas about the } x \text{ axis}}{\text{sum of the areas}}$$

$$= \frac{A_1y_1 + A_2y_2 + A_3y_3 + \cdots}{A_1 + A_2 + A_3 \cdots} \tag{34}$$

Example 2.—Find the center of gravity of an angle iron 6 by 4 by ½.

Solution.—Choose the outer legs *OB* and *OA* as the x and y axes. Divide the figure into two rectangles *OKEB*, with center at *I*, and *KAFC*, with center at *P* (Fig. 266).

$$\text{Let } A_1 = OKEB = 4 \times \tfrac{1}{2} = 2 \text{ sq. in.}$$
$$A_2 = KAFC = \tfrac{11}{2} \times \tfrac{1}{2} = 2.75 \text{ sq. in.}$$

Area	Moment A about y axis	Moment about x axis
$A_1 = 2$ $A_2 = 2.75$	$2 \times 2 = 4$ $2.75 \times \frac{1}{4} = 0.69$	$2 \times \frac{1}{4} = 0.5$ $2.75\left(\frac{1}{2} + \frac{11}{4}\right) = 8.94$

Then

$$\bar{x} = \frac{4 + 0.69}{2 + 2.75} = 0.99$$

$$\bar{y} = \frac{0.5 + 8.94}{2 + 2.75} = 1.99$$

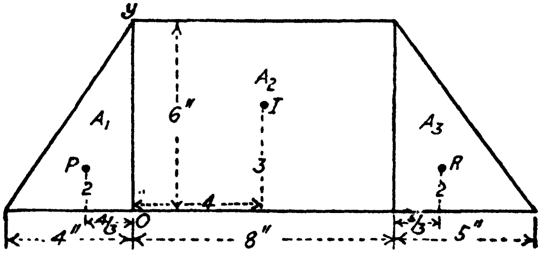


Fig. 267.

Note.—$\bar{y}$ could also have been obtained by extending *CF* to cut *OB* at point *D*. Then

$$\bar{y} = \frac{6 \times \frac{1}{2} \times 3 + \frac{7}{2} \times \frac{1}{2} \times \frac{1}{4}}{3 + 1.75} = 1.99$$

Example 3.—Find the center of gravity of three angle irons having the following dimensions, respectively: 3 by 4 by ⅜, 6 by 6 by ¾, 8 by 6 by 1. Verify the results from a handbook.

Example 4.—Find the center of gravity of the piece of sheet iron shown in Fig. 267.

Solution.—Divide the area into the three areas A_1, A_2, and A_3, with their centers as shown at P, I, and R. The moment arm for the point P with reference to the y axis is $-\frac{4}{3}$ since the point is in the second quadrant. The moment of A_1 would then be negative. This fact simply shows that area A_1 looked upon as a force tends to rotate about the y axis in the opposite direction to A_2 and A_3.

Area	x coor.	Ax	y coor.	Ay
$A_1 = 12$	$-\frac{4}{3}$	-16	2	24
$A_2 = 48$	4	192	3	144
$A_3 = 15$	$(8 + \frac{5}{3})$	145	2	30

Then

$$\bar{x} = \frac{-16 + 192 + 145}{12 + 48 + 15} = \frac{321}{75} = 4.28$$

$$\bar{y} = \frac{24 + 144 + 30}{12 + 48 + 15} = 2.64$$

Example 5.—Find the center of gravity of a plate 6 by 10 in. from which two circular rivet holes have been cut, as shown in Fig. 268.

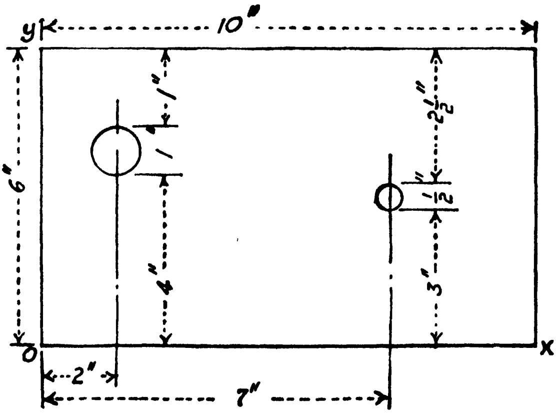


FIG. 268.

Solution.—In problems of this kind, treat the entire rectangular area as a force at its center, acting toward the center of the earth.

Then consider each circular area as a force at its center, acting in the opposite direction. The amount for each circle will then be negative.

Area	x coor.	y coor.	Ax	Ay
$6 \times 10 = 60$	5	3	300	180
$-\frac{\pi(1)^2}{4} = -\frac{\pi}{4}$	2	4.5	−1.57	−3.5343
$-\frac{\pi}{4}(\frac{1}{2})^2 = -\frac{\pi}{16}$	7	3.25	−1.37	−0.64

$$\bar{x} = \frac{300 - 1.57 - 1.37}{60 - (\pi/4) - (\pi/16)} = \frac{297.06}{59.02} = 5.04$$

$$\bar{y} = \frac{180 - 3.5343 - 0.64}{60 - (\pi/4) - (\pi/16)} = \frac{175.83}{59.02} = 2.98$$

Example 6.—An angle iron 3 by 3 by ½ in. is welded to a gusset plate, only side fillets being used. The angle iron is subjected to a pull of 27,000 lb. applied along the centroidal axis. What should be the respective lengths of the side fillets, if their lengths are inversely proportional to the distances from the centroidal axis?

Solution.—Taking moments about a point on P, we then have

$$Ly = L_1 y_1$$

The resistance of a ½-in. fillet weld is 4,000 lb. per in. of length. The total length required will be

$$L + L_1 = \frac{27{,}000}{4{,}000} = 6.75 \text{ in.}$$

From the properties of sections in Table XIII (Appendix) or a steel handbook,

$$y = 0.93, \quad y_1 = 2.07$$

Then

$$L \times 0.93 = (6.75 - L)2.07$$

Expanding and transposing produce

$$L \times 0.93 + L \times 2.07 = 6.75 \times 2.07$$

$$3L = 6.75 \times 2.07$$

$$L = \frac{6.75 \times 2.07}{3} = 4.66 \text{ in.}$$

$$L_1 = 6.75 - 4.66 = 2.09 \text{ in. } \textit{Ans.}$$

119. Center of Gravity of Weights and Volumes.—The weight of a body is a force acting at its own center of gravity and directed toward the center of the earth. The position of the center of bodies weighing W_1, W_2, W_3, etc., is found in the same manner as the resultant of parallel forces, *viz.*,

$$\left.\begin{aligned} \bar{x} &= \frac{\Sigma Wx}{\Sigma W} \\ \bar{y} &= \frac{\Sigma Wy}{\Sigma W} \\ \bar{z} &= \frac{\Sigma Wz}{\Sigma W} \end{aligned}\right\} \qquad \text{(I)}$$

If the bodies are all of the same material weighing w lb. per cu. ft., then

$$W_1 = wV_1, \qquad W_2 = wV_2, \qquad W_3 = wV_3, \text{ etc.}$$

Substituting in Eqs. (I) above, we have

$$\left.\begin{aligned} \bar{x} &= \frac{\Sigma wVx}{\Sigma wV_1} = \frac{\Sigma Vx}{\Sigma V} \\ \bar{y} &= \frac{\Sigma wVy}{\Sigma wV} = \frac{\Sigma Vy}{\Sigma V} \\ \bar{z} &= \frac{\Sigma wVz}{\Sigma wV} = \frac{\Sigma Vz}{\Sigma V} \end{aligned}\right\} \qquad \text{(II)}$$

That is, if the bodies are made of the same material and of the same density throughout, the center of gravity of bodies is their center of volume. If the bodies are of the same cross section but perhaps of different lengths,

$$V_1 = al_1, \; V_2 = al_2, \; V_3 = al_3, \text{ etc.}$$

Substituting in Eqs. (II) above produces

$$\left.\begin{aligned} \bar{x} &= \frac{\Sigma alx}{\Sigma al} = \frac{\Sigma lx}{\Sigma l} \\ \bar{y} &= \frac{\Sigma aly}{\Sigma al} = \frac{\Sigma ly}{\Sigma l} \\ \bar{z} &= \frac{\Sigma alz}{\Sigma al} = \frac{\Sigma lz}{\Sigma l} \end{aligned}\right\} \qquad \text{(III)}$$

If the bodies are parts of a wire or slender rod of negligible cross section, then their center of gravity may be found from the center of their lengths. *The center of gravity of a cylinder of revolution is at the center of its geometric axis.*

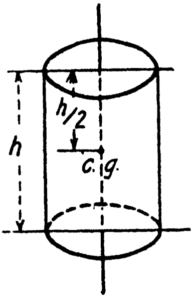


FIG. 269.

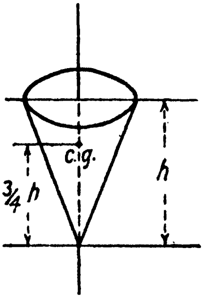


FIG. 270.

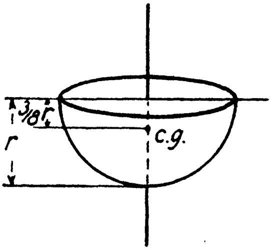


FIG. 271.

The center of gravity of a cone of revolution is on the geometric axis, $\frac{1}{4}h$ *from the base or* $\frac{3}{4}h$ *from the vertex, where* h *is the altitude of the cone.*

The center of gravity of a hemisphere of radius r *is* $\frac{3}{8}r$ *from the center of the sphere.*

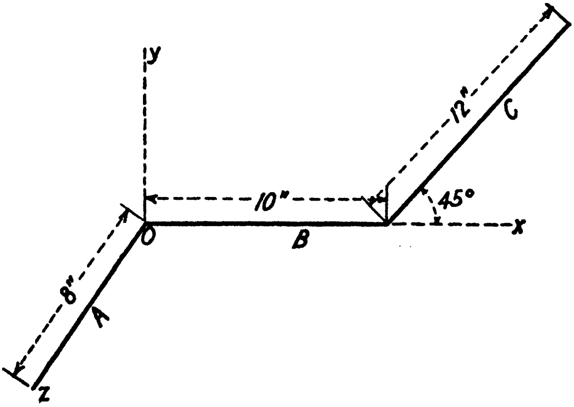


FIG. 272.

Example 1.—A thin wire is bent into a shape shown in Fig. 272. Find the coordinates of its center of gravity.

Solution.—Since lengths are involved, Equations (III) above will be used. It will be seen that for A, the center of the 8-ft. length,

$x = 0$, $y = 0$, $z = 4$; for B, $x = 5$, $y = 0$, $z = 0$; for C, $x = 10 + 6 \cos 45 = 14.2$, $y = 6 \cos 45 = 4.2$, $z = 0$.

l	x	y	z	lx	ly	lz
8	0	0	4	0	0	32
10	5	0	0	50	0	0
12	14.2	4.2	0	170.4	50.4	0
30				220.4	50.4	32

Also,

$$\Sigma l = 30, \qquad \Sigma lx = 220.4, \qquad \Sigma ly = 50.4, \qquad \text{and } \Sigma lz = 32$$

$$\bar{x} = \frac{220.4}{30} = 7.35, \qquad \bar{y} = \frac{50.4}{30} = 1.68 \text{ ft.}, \qquad \bar{z} = \frac{32}{30} = 1.07 \text{ ft.}$$

Example 2.—A cylinder for which $r = 2$ in. and $h = 6$ in. has an inverted cone cut from it (Fig. 273). The cone, for which $r = 2$ in. and $h = 3$ in., has its base in the upper base of the cylinder. Find the center of gravity of the part remaining.

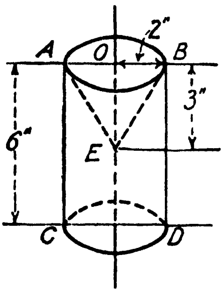


FIG. 273.

Solution.—Choose 0 as the origin. Since the figure is symmetrical with respect to the axis OE, the center is on this axis and we need find only y.

The volume of the cylinder $V_1 = \pi \times 4 \times 6 = +24\pi$ and for the cone $V_2 = -\frac{1}{3}\pi \times 4 \times 3 = -4\pi$.

V_2 is negative because it is volume removed.

Then, from Eq. (II) above,

$$\bar{y} = \frac{\Sigma Vy}{\Sigma V} = \frac{24\pi + \frac{6}{2} - 4\pi(\frac{1}{4} \times 3)}{24\pi - 4\pi}$$

$$= \frac{72 - 3}{20} = 3.45 \text{ in.}$$

Example 3.—Suppose the cylinder of Example 2 is made of cast iron weighing 450/1,728 = 0.23 lb. per cu. in. and, after the conical hole is bored out, it is filled with concrete which weighs 150/1,728 = 0.08 lb. per cu. in. Find the center of gravity of the solid.

Solution.—The weight of the cylinder $W_1 = 24\pi \times 0.23$ lb. The weight of the cone $W_2 = -4\pi \times 0.15$ lb. That is, we subtract $4\pi \times 0.23$ and add $4\pi \times 0.08$ which gives the actual quantity as above.

Then

$$\bar{y} = \frac{\Sigma Wy}{\Sigma W} = \frac{24\pi \times 0.23 \times 3 - 4\pi \times 0.15 \times \frac{3}{4}}{24\pi \times 0.23 - 4\pi \times 0.15} = 3.27 \text{ in.}$$

120. Areas and Solids by Revolution.—Since the center of gravity of an area or a body is the point at which the area or mass of the body may be assumed to be concentrated, it can be said *that the distance through which an area or a body moves is the same as the distance described by its center of gravity.* This relation is used in finding areas and volumes.

Thus, a line moving parallel to its original position is said to generate an area *that is equal to the length of the line multiplied by the distance through which its center of gravity moves.* That is, area is equal to length times width.

Also, an area moving parallel to its original position is said to develop the volume of a prism *that is equal to the area of a rectangle multiplied by the distance through which the center of gravity moves.* That is, a volume is equal to the area of the base times the altitude.

Similarly, a line rotating about one end will develop the area of a circle. A right triangle rotating about either leg will develop the volume of a cone. In each case a line or an area moves through a distance equal to the length of a path described by the center of gravity of either the line or the area. Many determinations of areas or volumes are simplified by the use of this method.

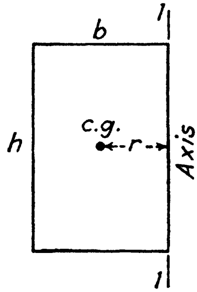


FIG. 274.

Example 1.—Find the volume of a vertical cylinder by the method of revolving an area (Fig. 274).

Solution.—The vertical section of the cylinder is a rectangle. It may be seen that the volume of this cylinder can be developed by revolving the rectangle about a vertical edge. Then

$$\text{Area} = bh$$

where b = radius of cylinder = width,
h = height.

Distance from edge to center of gravity equals $b/2$. In one revolution, center of gravity moves a distance

$$d = 2\pi \times \frac{b}{2}$$

Then

$$\text{Volume} = bh \times \frac{2\pi b}{2} = \pi b^2 h$$

This result is the same as that for volume determined by the ordinary methods of mensuration, where πb^2 is the area of the base and h is the altitude.

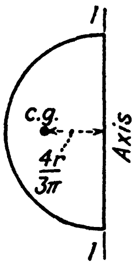


Fig. 275.

Problems

207. Find the volume of a cone by the method of revolving a right triangle with a 3-in. base and 4-in. altitude about the vertical edge.

208. Find the volume of a hollow cylinder by the method described in Art. 120.

209. Find the volume of a sphere by revolving a semicircle about a diameter (Fig. 275).

210. Find the volume of the rim of a flywheel 6 ft. in diameter, if the rim is 12 in. wide and 8 in. thick.

121. Stability of a Body.—The stability of an object depends on the position of its center of gravity. If the vertical line through the center of gravity cuts the base, the body is stable. If it falls outside the base, the body is in an unstable condition. If any body is tipped until the vertical, or plumb line, through the center of gravity falls outside the base, the body will tip over without an extra force. Heavy bodies with large base and the center of gravity low are less likely to tip over than if the base were narrow and center of gravity high.

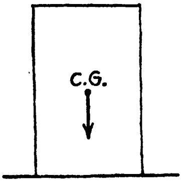


Fig. 276.

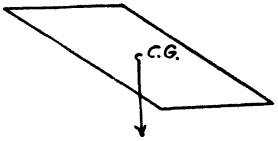


Fig. 277.

In Fig. 276, the rectangle is stable; in Fig. 277, the parallelogram is unstable.

In order to illustrate the preceding statements, Fig. 278 is shown, representing a section through the center of gravity of a box weighing W lb. In the left-hand figure the box is tilted on the corner A. There are two forces W as shown when the lifting force is removed. They form a couple of moment $a \times W$ that rotates the box to the left and therefore brings it to its original position.

In the middle figure the box is tilted until the center of gravity is in the same vertical line with the reaction at A. The moment of the couple is zero.

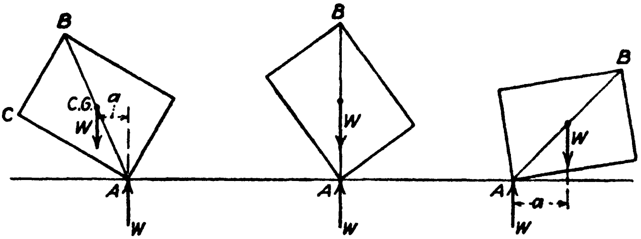


FIG. 278.

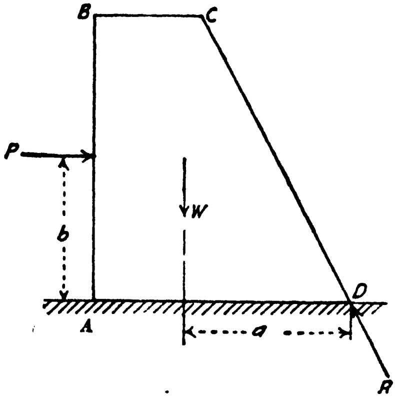


FIG. 279.

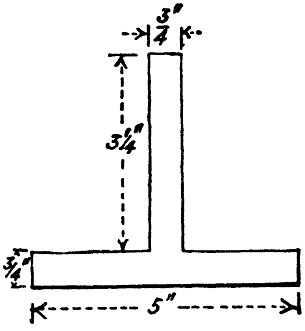


FIG. 280.

In the right-hand figure the vertical force W through the center of gravity now falls without the original base AC. The moment aW is such as to continue the rotation without any other applied force.

122. Stability against Rotation.—Let Fig. 279 be the section of a retaining wall, W lb. its weight, and P a force tending to rotate the wall. Now, if the resultant of P and W falls to the left of D, the wall is stable. If it passes through D, the wall is neutral. If to the right of D, the wall is unstable.

Algebraically, the solution consists in taking moments around point D. The following conditions result: If

$$Pb < Wa \text{ (stable)}$$

and

$$Pb = Wa \text{ (neutral)}$$

and

$$Pb > Wa \text{ (unstable)}$$

Then, to find the force P that would upset the wall, solve $Pb = Wa$ for P. Any value greater than the one thus found would be dangerous.

The stability of the wall may be increased by taking a greater weight or distributing the material so that the center of gravity has a longer moment arm.

Example 1.—A car 28 ft. long, 8 ft. wide, and 8 ft. high weighs 10 tons. The car floor is 3 ft. above the track. The rails are 5 ft. from center to center. Find the wind pressure that would upset the car.

Solution.—Let p be the wind pressure per square foot of surface, at right angles to the side of the car. Since the car surface is $28 \times 8 = 224$ sq. ft., the total pressure is $224p$ lb., and it will act at the center of the side. The moment arm is $3 + 4 = 7$ ft. The moment is $7 \times 224p = 1{,}568\ p$.

The moment of the weight is $20{,}000 \times 2\frac{1}{2} = 50{,}000$.

Equating moments,

$$1{,}568p = 50{,}000$$
$$p = 31.9 \text{ lb./sq. ft.}$$

Verify this result graphically.

123. Dams.—In the design and construction of a dam or a retaining wall there are three typical computations: (*a*) to find the factor of safety against overturning, (*b*) to find whether or not the dam is safe against sliding, and (*c*) to find where the resultant of the water pressure and the weight of the dam, or the resultant of the earth pressure and the weight of the wall, intersects the base of the dam or the wall. This last computation is used further in determining the compressive stresses in the foundation.

To illustrate the computations, let Fig. 281 represent a dam of dimensions as shown. The water pressure will tend to overturn the dam about the toe A. The weight of the masonry acting through the center of gravity of the cross section will resist the overturning. The intensity of the water pressure varies with the depth. If w is taken as the weight of 1 cu. ft. of water, then wh is the pressure in pounds per square foot at a depth h, and $wh/2$ will be the average pressure in pounds per square foot against the dam.

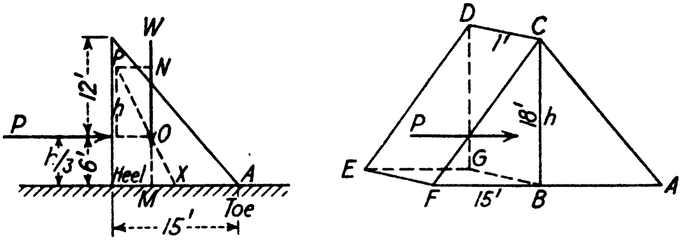


FIG. 281.

The total pressure against the dam per foot of length will be equal to

$$P = \frac{wh}{2} \times h \times 1 = \frac{wh^2}{2} \qquad (35)$$

But this force may be regarded as equivalent to the volume of a solid, as will now be shown. In the figure, FB and EG represent the intensity of pressure at depth h; then $CDEFBG$ is a wedge in which the dimension CD is taken as 1 ft. Then the volume of this wedge is equal to

$$FB \times BC \times \tfrac{1}{2} \times CD$$

But $FB = wh$ and $BC = h$, and
volume $= (wh) \times (h) \times (1) \times (\tfrac{1}{2}) = wh^2/2$ as above.

The solid that may be used as the basis for determining the value of a force is called the *pressure solid.* In the case just computed it is known as the *pressure wedge.*

Since the total pressure of the water against the dam per foot of length is equivalent to the volume of a solid, then the line of action of the corresponding force will pass through the center of gravity of the solid. Thus with reference to Fig. 281, it will pass through a point which is two-thirds of the height *BC* below the line *CD*. That is, the force acts through the center of gravity of a wedge and in a horizontal direction. The section of the dam is also a wedge, and its weight acts vertically through the center of gravity. Then

$$\text{Moment of water pressure} = \frac{wh^2}{2} \times \frac{h}{3} = \frac{wh^3}{6} = \text{overturning moment}$$

$$\text{Moment of weight of dam} = w'(bh) \times \left(\frac{1}{2}\right) \times (1) \times \left(\frac{2}{3}b\right) = \frac{wb^2h}{3}$$

In a correctly proportioned dam, the resisting moment is twice the overturning moment.

Example 1.—Using the dimensions of the dam of Fig. 281, find the point at which the resultant of the forces acting on the dam intersect the base of the dam. If a coefficient of friction between the dam and its base may be taken as 0.7, is the dam safe against sliding?

Solution.

$$\text{Water pressure} = \frac{wh^2}{2} = \frac{62.5 \times 18^2}{2} = 10{,}125 \text{ lb.}$$

$$\text{Weight of dam} = 150 \times 18 \times 15 \times \tfrac{1}{2} = 20{,}250 \text{ lb.}$$

Construct the force parallelogram as shown. Then *OP* is the resultant which intersects the base at *X*. Triangle *ONP* is similar to triangle *OMX*. By proportion,

$$\frac{MX}{OM} = \frac{PN}{ON}, \qquad OM = 6 \text{ ft.}$$

$$\frac{MX}{6} = \frac{10{,}125}{20{,}250}, \qquad MX = \frac{6 \times 10{,}125}{20{,}250} = 3 \text{ ft.}$$

Since *BM* is equal to one-third of *BA*, or 5 ft., then the resultant intersects the base at a distance of 8 ft. from *B* or 7 ft. from *A*.

To determine whether the dam is safe against sliding, it is necessary to compare the water pressure with the friction between the dam and its base. Thus,

$$\text{Water pressure} = 10{,}125 \text{ lb.}$$

$$\text{Friction} = 0.7 \times 20{,}250 = 14{,}175 \text{ lb.}$$

The factor of safety

$$F = \frac{14,175}{10,125} = 1.4 \text{ } \textit{Ans.}$$

From the answer, above, it is seen that point X in part (a) (Fig. 281) is within the middle one-third of the base. This position for X is considered good practice. However, it is sufficient for this point to fall inside of the middle one-third and adjacent to the one-third point nearest to the toe. In a later chapter, the reasons for the foregoing statements will be given and explained.

Problems

211. Find the center of gravity of the section of the T beam shown in Fig. 280.

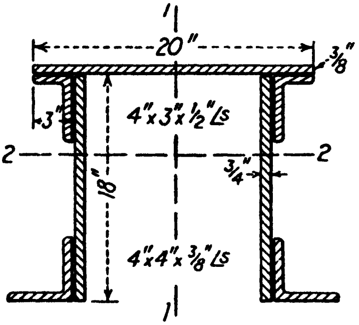


FIG. 282.

212. Locate the center of gravity of the channel shown in Fig. 283.

213. In Fig. 279, AB = 16 ft., BC = 6 ft., AD = 16 ft. With W = 1,000 lb. and P = 800 lb., find how high above A the force P may be applied to be on the point of upsetting the figure.

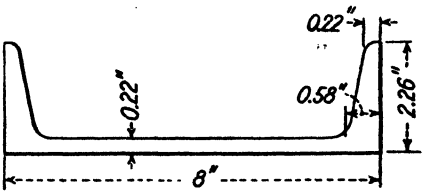


FIG. 283.

214. A retaining wall 15 ft. long has the cross section described in Prob. 213. The material weighs 150 lb. per cu. ft. Apply a force P equal to 100,000 lb. at heights of 4 ft., 5 ft., and 6 ft. above A. In each case, find whether the wall is stable or unstable.

215. In Example 1 (Art. 122), if the car is 36 ft. long, and the wind pressure is 50 lb. per sq. ft., will the car upset?

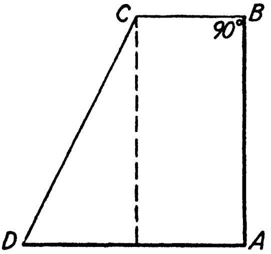


Fig. 284.

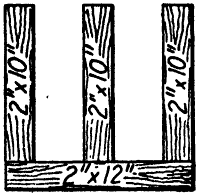


Fig. 285.

216. A standpipe 60 ft. high and 16 ft. outside diameter is filled with water. It must withstand a wind pressure of 40 lb. per sq. ft. on the vertical projection of its curved surface. What is the factor of safety against overturning?

217. Three 2- by 10-in. planks are nailed to a 2- by 12-in. plank as shown in Fig. 285. How far is the center of gravity above the base?

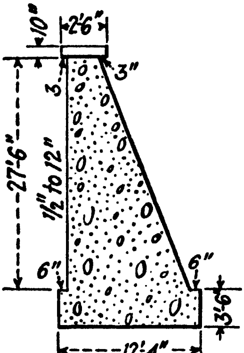
218. Find the center of gravity of a board 10 ft. long, 12 in. wide at one end, and 6 in. at the other. Hint.—Complete the triangle.

219. Find the center of gravity of the section of a bridge member made of two 18- by ¾-in. web plates, a 20- by ⅜-in. top plate, two top angles 4 by 3 by ½ in., and two bottom angles 4 by 4 by ⅜ in. (Fig. 282).

220. In Fig. 284, $AB = 16$ ft., $BC = 6$ ft., $AD = 12$ ft. Find the coordinates of the center of gravity, using A as a reference point.

221. A cylinder 1 ft. in diameter, 4 ft. high, and weighing 400 lb. stands on end. What force applied at the middle of the height would be on the point of overturning the cylinder? Check this result graphically.

222. Figure 286 shows a cross section of a retaining wall. The batter of one face is ½ in. in 12 in. In order to test for stability, it is necessary first to find where the line of action of the resultant weight cuts the base. Locate this point.



Subway Retaining Wall

Fig. 286.

223. In problem 222, the wall weighs 150 lb. per cu. ft. Assume a portion of the wall 1 ft. long. If a horizontal earth pressure of 30,000 lb. acts 10 ft. above the base, find where the resultant force on the wall intersects the base.

224. Figure 287 shows a trap door in the bottom of a bucket used in handling concrete. The bucket is built of $\frac{1}{4}$-in. steel plate. The total depth is 4 ft. 6 in. If the bucket contains concrete as shown, how far will the center of gravity of the bucket and its contents be above the bottom?

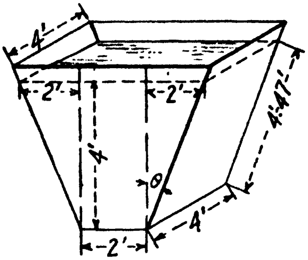


FIG. 287.

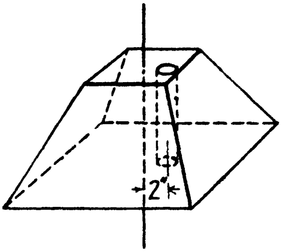


FIG. 288.

225. A dam stands 12 ft. high. It is 4 ft. wide at the top and 10 ft. wide at the base. The upstream face is vertical. The coefficient of friction between the dam and its foundation is 0.6. What are the factors of safety against sliding and overturning when the water has reached the top of the dam?

226. The frustum of a regular pyramid has a lower base 12 in. square and upper base 6 in. square. The altitude of the frustum is 6 in. A 1-in. hole is drilled, as shown in Fig. 288, 2 in. from the vertical axis of the frustum and in a direction along the diagonal of the base. Find the position of the center of gravity.

227. Locate the center of gravity of the solid shown in Fig. 289.

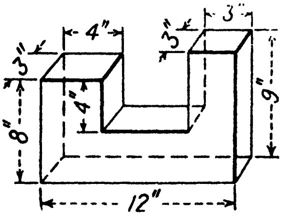


FIG. 289.

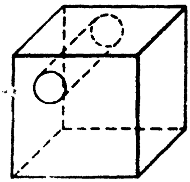


FIG. 290.

228. A 6-in. cube of wood weighing 40 lb. per cu. ft. has a hole 1 in. in diameter drilled through it parallel to one edge (Fig. 290). The long axis of the hole is 2 in. from the upper surface and 2 in. from the left side of the cube. Locate the center of gravity. If this hole is filled with lead weighing 710 lb. per cu. ft., where is the center of gravity?

229. Locate the center of gravity of Fig. 291 with respect to the upper right-hand corner.

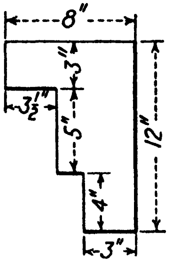


Fig. 291.

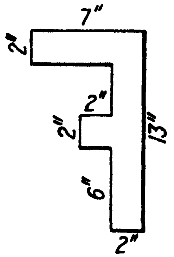


Fig. 292.

230. How far is the center of gravity from the long edge of Fig. 292?

231. In Fig. 293, how far is the center of gravity above the lower edge?

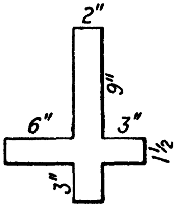


Fig. 293.

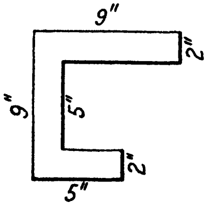


Fig. 294.

232. Locate the center of gravity of Fig. 294 with reference to axes through the lower left-hand corner of the figure.

CHAPTER XI

MOMENT OF INERTIA

124. Theory.—Figure 295 is a rectangle *CD*, with a line *AB* parallel to the base, passing through the center of gravity. The figure is divided into small rectangles as shown. The areas are represented by a small *a*, with a subscript. The distance from the axis *AB* to the center of gravity of these rectangles is represented by *y* with a subscript, corresponding to the area. Thus for area a_1, there is a moment arm y_1, etc. If now each area is multiplied by the square of its moment arm and the sum of all such terms taken, as the number of small areas into which the rectangle *CD* is divided becomes very great, the result is called ***the moment of inertia of the rectangle about the gravity axis AB.*** The equation is

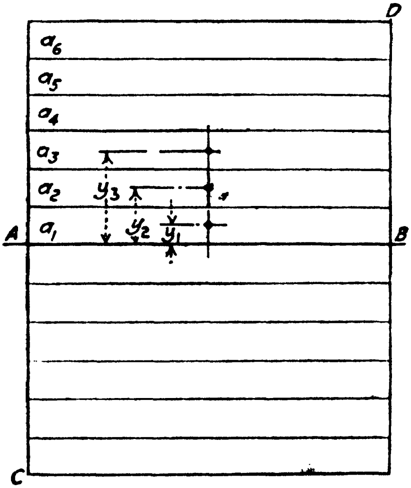


FIG. 295.

$$I_0 = lim\ \Sigma a_r y_r^2 \tag{36}$$

where I_o stands for the moment of inertia about a gravity axis.

The calculation of I_o is a problem of calculus and is beyond the scope of this work. Fortunately, the areas involved in engineering problems are rectangles, triangles, circles or a combination of one or more of these figures, in the majority of cases, and the computation of I is not a difficult problem.

By means of the calculus, the following formulas have been derived:

The moment of inertia of a rectangle about a gravity axis parallel to the base, such as AB of Fig. 296 is

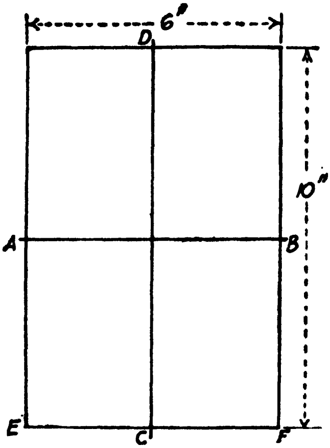


FIG. 296.

$$I_o = \tfrac{1}{12}bh^3 \qquad (37)$$

where b is the base and h the altitude of the rectangle.

The I for a rectangle about a base EF is

$$I = \tfrac{1}{3}bh^3 \qquad (38)$$

The I_o for a triangle about a gravity axis parallel to the base is

$$I_o = \tfrac{1}{36}bh^3 \qquad (39)$$

The I_o for a circle about a diameter is

$$I_o = \frac{\pi}{4}r^4 \qquad (40)$$

Example 1.—Find the moment of inertia of the rectangle shown in Fig. 296 about gravity axes AB and CD.

Solution.—b is the dimension parallel to the axis, and h the dimension perpendicular to the axis. Then, $b = 6$, and $h = 10$.
Therefore, by Eq. (37),

$$I_0 \text{ about } AB = \frac{1}{12}bh^3 = \frac{6 \times 10^3}{12} = 500 \text{ in.}^4$$

For the axis CD, $b = 10$, and $h = 6$.
Therefore, by Eq. (37),

$$I_0 \text{ about } CD = \frac{1}{12}bh^3 = \frac{10 \times 6^3}{12} = 180 \text{ in.}^4$$

Moments of inertia are expressed in inches to the fourth power, sometimes called *biquadratic inches.*

The results of the previous example show that the moment of inertia of the rectangle is greater about the axis AB than around or with reference to axis CD. The moment of inertia is then seen to depend on the arrangement of the area with reference to the axis. The farther the area is from the axis, the greater I becomes. This principle is very important in designing beams and columns.

As a simple experiment, take a 1-in. board with dimensions in Fig. 296. Spin the board around AB and then around CD. It will be found that it requires a much greater force to rotate the board around AB than around CD. Conversely, if the board is already rotating, a greater force is necessary to stop rotation around AB than around CD.

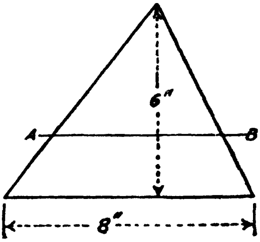


Fig. 297.

Example 2.—Find I_0 of the triangle with reference to axis AB shown in Fig. 297.

Solution.—$b = 8$, and $h = 6$.
Then, by Eq. (39),

$$I_0 = \frac{1}{36}bh^3 = \frac{8 \times 6^3}{36} = 48 \text{ in.}^4$$

Example 3.—Find I_0 for a circle of radius 4 in.

Solution.—$r = 4$.
Then, by Eq. (39),

$$I_0 = \frac{\pi}{4}r^4 = \frac{\pi \times 4^4}{4} = 64\pi$$

Example 4.—Find I for each of the gravity axes parallel to the sides of three rectangles having the following dimensions: 4 by 10 in., 8 by 12 in., 10 by 10 in.

125. Illustrations.—Take a rectangle 6 by 8 in. Divide it into eight rectangles, each 1 in. high. Multiply each area by the square of the distance of its center of gravity from *AB*, and find the sum of such terms (Fig. 298).

$$\text{Sum} = (6 \times 0.5^2 + 6 \times \overline{1.5}^2 + 6 \times \overline{2.5}^2 + 6 \times \overline{3.5}^2)2 = 252$$

The sum of the four rectangles above *AB* is multiplied by 2, for there are as many equal rectangles below the axis as above.

Now, divide each rectangle into two rectangles, and show that the sum obtained by multiplying each area by the square of the distance of its center of gravity from *AB*, and adding these terms, is 255. The more rectangles we use, the greater the sum becomes; but it is always less than 256.

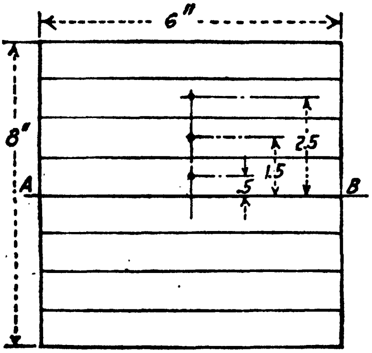


FIG. 298.

But, by Eq. (37),

$$I_{AB} = \frac{1}{12}bh^3 = \frac{6 \times 8^3}{12} = 256 \text{ in.}^4$$

This problem illustrates the truth of the definition that the moment of inertia is the limit of the sum of a very large number of terms obtained by multiplying each small area by the square of the distance of its center of gravity from the axis.

126. Radius of Gyration.—The radius of gyration is the distance from the axis of a plane area to a place where all the area may be concentrated and the moment of inertia is left unchanged.

Then, by definition,

$$I = Ar^2 \qquad (41)$$

where r is the radius of gyration.

In the last example solved, $I = 256$, and $A = 48$.

$$\therefore 256 = 48r^2$$
$$r^2 = \frac{256}{48} = 5.33, \qquad r = 2.31$$

A flywheel affords a good illustration of the radius of gyration. By putting all the material (hub and spokes being neglected) in the rim, a greater moment of inertia is obtained than if the same material were made into a solid wheel. The radius of gyration of a flywheel is taken as the mean radius. The moment of inertia of a flywheel is then the mass of the rim, times the square of the mean radius.

Example.—Find the radius of gyration of an 8- by 10-in. rectangle with reference to each gravity axis parallel to the sides.

127. Transfer Formula.—It is often desirable to obtain the moment of inertia of an area about some axis not passing through the center of gravity of the area. With reference to Art. 124, if the distances from the small rectangles are measured from some axis parallel to AB, then each value of y is increased by an amount d, where d is the distance between the gravity axis AB and the parallel axis. Equation (36) will be modified into the form

$$I_x = \lim \Sigma a_r(y_r + d)^2$$
$$I_x = \lim \Sigma a_r(y_r^2 + 2y_rd + d^2)$$
$$I_x = \lim \Sigma a_ry_r^2 + \lim \Sigma a_rd^2$$

or

$$I_a = I_0 + Ad^2 \tag{42}$$

The middle term above will vanish because it represents the moment of the area about its own gravity axis. In the formula thus obtained, I_0 is with respect to a gravity axis, I_a is with respect to any axis parallel to the gravity axis, d is the distance between the parallel axes, and A is the area of the figure. The formula is spoken of as a transfer formula. NOTE.—*The transfer must be to or from a parallel gravity axis.*

Example 1.—Find the moment of inertia of a 6- by 10-in. rectangle about its 6-in. base (Fig. 299).

Solution.—Let AB be a gravity axis parallel to CD the base.

$$I_0 = \frac{1}{12}bh^3 = \frac{6 \times 10^3}{12} = 500$$

$$d = 5 \text{ (the distance between the axis)}.$$

Then, by Eq. (42),

$$I_a = I_0 + Ad^2 = 500 + 60 \times (5)^2 = 2{,}000 \text{ in.}^4 \tag{42}$$

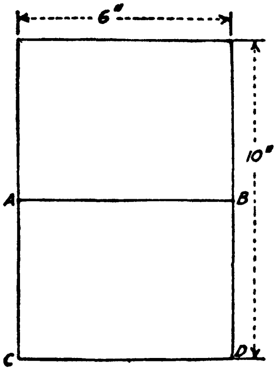


FIG. 299.

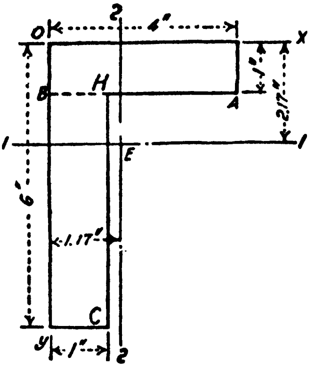


FIG. 300.

Example 2.—Find the moment of inertia of a rectangle about its base. Let b be the base and h the altitude. Then I_0 being given.

$$I_0 = \frac{1}{12}bh^3, \qquad \text{and } d = \frac{h}{2}$$

$$\begin{aligned} I_a &= I_0 + Ad^2 \\ &= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 \\ &= \frac{1}{12}bh^3 + \frac{bh^3}{4} = \frac{bh^3}{3} \end{aligned}$$

In the preceding example, $b = 6$ and $h = 10$.

$$\therefore\ I_a = \frac{1}{3}bh^3 = \frac{6 \times 10^3}{3} = 2{,}000 \text{ in.}^4$$

Example 3.—Show that the moment of inertia of a triangle about its base is $\frac{1}{12}bh^3$; that the moment of inertia of a triangle about a parallel axis through the vertex is $\frac{1}{4}bh^3$, I_0 being given.

Example 4.—Find the moment of inertia of a 6- by 4- by 1-in. angle iron about the gravity axes parallel to the two legs.

Solution.—Choose O as an origin, and the two legs as the x and y axes (Fig. 300). Let E be the center of gravity, $1 - 1$ and $2 - 2$ the gravity axes with reference to which the moment of inertia is required. From a handbook, E is found to be 1.17 in. from the y axis and 2.17 from the x axis. These results may be verified by the methods described in the previous articles on center of gravity.

Divide the angle into two rectangles, OA and BC, each of which has its base on the common axis Oy, and find I about the y axis, the base for each rectangle.

$$\begin{aligned} I_a \text{ for } OA &= \tfrac{1}{3}bh^3 = \tfrac{1}{3} \times 1 \times 4^3 = \tfrac{64}{3} \\ I_a \text{ for } BC &= \tfrac{1}{3}bh^3 = \tfrac{1}{3} \times 5 \times 1^3 = \tfrac{5}{3} \end{aligned}$$

Since I_a for each rectangle is referred to a common base, they may be combined. Then, for the entire angle,

$$I_a = \tfrac{64}{3} + \tfrac{5}{3} = \tfrac{69}{3} = 23$$

Now, I_{2-2} or the I_0 desired is with reference to the gravity axis of the angle. $d = 1.17$, $A = 4 \times 1 + 5 \times 1 = 9$, and $I_0 = I_{2-2}$. Then from

$$\begin{aligned} I_a &= I_0 + Ad^2 \\ 23 &= I_{2-2} + 9 \times (1.17)^2 \\ I_{2-2} &= 23 - 12.32 = 10.68 \end{aligned}$$

Another method is to calculate I_0 for each rectangle separately with reference to a gravity axis parallel to $2 - 2$ and then to transfer each to $2 - 2$, as follows:

$$I_0 \text{ for } OA = \frac{1}{12}bh^3 = \frac{1 \times 4^3}{12} = \frac{64}{12}$$

But the center of gravity of the rectangle OA is 2 in. from the y axis. Then d, the distance to the parallel axis, is $d = 2 - 1.17$. Then

$$\begin{aligned} I_a = I_0 + Ad^2 &= \tfrac{64}{12} + 4(2 - 1.17)^2 = 8.09 \\ I_0 \text{ for } BC &= \frac{1}{12}bh^3 = \frac{5 \times 1^3}{12} = \frac{5}{12} \\ I_a &= \tfrac{5}{12} + 5(1.17 - 0.5)^2 = 2.667 \\ I_{2-2} &= 8.09 + 2.667 = 10.757 \text{ in.}^4 \end{aligned}$$

The student will notice that the value of I_{2-2} obtained by the two methods are not quite the same. This difference is due to the value 1.17 not being carried to the third decimal.

As an exercise the student is asked to calculate I_{1-1} and find it equal to 30.75.

Hint.—Produce *CH* to meet *OX*, and work with the two rectangles formed.

Example 5.—Find I_{1-1} and I_{2-2} for two angle irons: 4- by 4- by ½-in.; 4- by 3- by ¾-in. Compare results with the values given in the handbook.

Example 6.—Find I_{1-1} and I_{2-2} for a girder made with four 4- by 3½- by ⅜-in. angle irons and one 12- by ⅜-in. web plate (Fig. 301).

Solution.—From a handbook, the area of a 4 by 3½ by ⅜ angle iron is 2.68, and the distance from the back of the long leg to the center of gravity is 0.96. Also, the moment of inertia with reference to a gravity axis through *C*, parallel to 2 – 2, is 2.99.

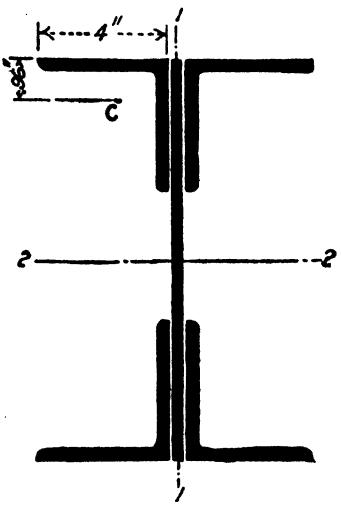


Fig. 301.

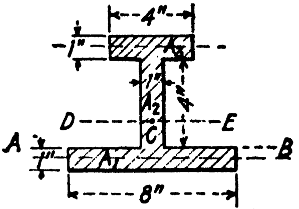


Fig. 302.

Then, for each angle iron,

$$I_{2-2} = I_0 + Ad^2 = 2.99 + 2.68 \times (6 - 0.96)^2 = 71.06$$

For all four angles,

$$I_{2-2} = 4 \times 71.06 = 284.24$$

For the web plate,

$$I_{2-2} = \tfrac{1}{12}bh^3 = \tfrac{1}{12} \times \tfrac{3}{8} \times \overline{12^3} = 54$$

Since the moments of inertia of the several parts are referred to a common axis 2 – 2, they may be added.

$$\therefore\ I_{2-2} \text{ for the section } = 284.24 + 54 = 338.24 \text{ in.}^4$$

The value of I_{1-1} is left for the student to calculate.

Example 7.—Find I_{1-1} and I_{2-2} for a girder built up of four 5- by 3½- by ½-in. angle irons and a 10- by ½-in. web plate.

Example 8.—Find *I* about the gravity axis parallel to the base of the area shown in Fig. 302.

Solution.—First find the center of gravity of the area, using AB as the axis of reference.

Since the area is symmetrical, the only calculation necessary is for $\bar{y}$.

$$\bar{y} = \frac{-8 \times \frac{1}{2} + 4(2) + 4 \times (4.5)}{8 + 4 + 4} = 1.375$$

a. First find the moment of inertia about the axis AB.

$$\begin{aligned} I_{AB} \text{ for } A_1 &= \tfrac{1}{3}bh^3 = \tfrac{1}{3} \times 8 \times 1 = 2.67 \\ I_{AB} \text{ for } A_2 &= \tfrac{1}{3}bh^3 = \tfrac{1}{3} \times 1 \times 64 = 21.33 \\ I_{AB} \text{ for } A_3 &= \tfrac{1}{12} \times 4 \times 1 + 4 \times \overline{4.5}^2 = \underline{81.33} \\ \text{Total } I \text{ for area} &= 105.33 \end{aligned}$$

Now, transfer from axis AB to a parallel axis DE through the center of gravity C.

$$\begin{aligned} 105.33 &= I_{DE} + 16(1.37)^2 \\ I_{DE} &= 75.07 \text{ in.}^4 \end{aligned}$$

b. Another method is as follows: The I for area may be obtained by getting I for each rectangle separately about its gravity axis parallel to DE and then transferring to DE.

$$\begin{aligned} I_{DE} \text{ for } A_1 &= \tfrac{1}{12} \times 8 \times 1 + 8(1.87)^2 = 28.79 \\ I_{DE} \text{ for } A_2 &= \tfrac{1}{12} \times 1 \times 64 + 4(0.63)^2 = 6.89 \\ I_{DE} \text{ for } A_3 &= \tfrac{1}{12} \times 4 \times 1 + 4(3.13)^2 = \underline{39.39} \\ \text{Total } I_{DE} &= 75.07 \text{ in.}^4 \end{aligned}$$

128. Principal Axes.—For the rectangle, moments of inertia with respect to the axis of symmetry represent maximum and minimum values and are known as *principal moments of inertia.* Advantage is taken of this fact in the design of beams by placing the long axis in the vertical position. Most of the sections in actual use have either one or two axes of symmetry. The gravity axes that are also axes of symmetry are called *principal axes.* If there is only one axis of symmetry, as in the T beam, then this axis together with the gravity axis perpendicular to it constitute the principal axes. These axes and the corresponding moments of inertia are always involved in beam and column design.

In the handbooks, principal axes are usually referred to as axes 1-1 and 2-2 and sometimes as x-x and y-y axes. The plate and angle section in Fig. 301 shows the principal axes.

129. Built Sections.—When the moment of inertia of a girder section is required that has been made through the center of rivet holes, proceed as follows: (*a*) Find the moment of inertia of the section without any reference to the rivet hole. (*b*) Find the moment of inertia of the rivet hole with reference to the required axis. The difference between the two results is the required moment of inertia.

In general, the determination of the moment of inertia of an area with respect to the gravity axes is one of the important steps in the design of beams, columns, and shafts.

Problems

233. Find the moments of inertia with respect to both principal axes (Fig. 280).

234. In Fig. 266, find the moment of inertia with respect to the horizontal gravity axis and also with respect to the left edge of the figure.

235. What is the moment of inertia with respect to the horizontal gravity axis of the column section shown in Fig. 303? The area of a Z bar is 8.63 sq. in., and the moment of inertia of a Z bar with respect to its gravity axis parallel to the shorter leg is 42.12.

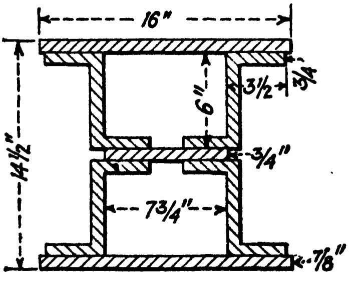


FIG. 303.

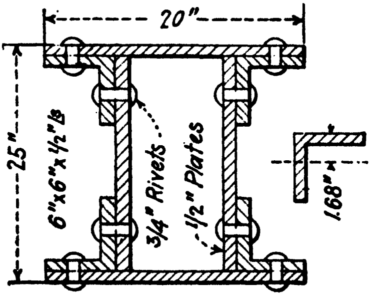


FIG. 304.

236. A box girder is shown in Fig. 304. What is the moment of inertia with respect to the horizontal gravity axis if the rivet holes are deducted? The area of an angle is 5.75 sq. in. The moment of inertia of an angle with respect to its gravity axis parallel to either leg is 19.91.

237. Find the moment of inertia of the section shown in Fig. 282 with respect to both principal axes.

238. What are the moments of inertia with respect to the principal axes in Fig. 285?

239. Show two ways in which it is possible to compute the moment of inertia of the T section shown in Fig. 280 with respect to the horizontal and vertical gravity axes.

CHAPTER XII

BEAMS AND BEAM DESIGN

130. Elements of Theory.—Let Fig. 305 represent a simple beam supported at both ends, carrying a load W at its center. Pd is the center line of the top of the beam on which the load is concentrated. The load produces a bending in the beam at all points. The assumption is made that any rectangular sections such as $AEFB$, $HLMI$ remain rectangles. But they are rotated into the positions $A'E'F'B'$, and $H'L'M'I'$ by the bending of the beam. The length EL is shortened to $E'L'$ or compressed, and a length AH is lengthened to $A'H'$. The resistance offered to this shortening and lengthening of the fibers is called *internal fiber stress* or simply *internal stress.* The upper fibers are in compression and the lower ones in tension.

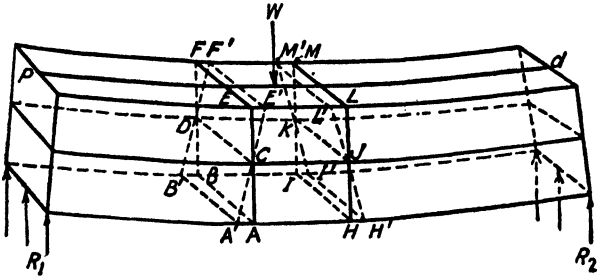


Fig. 305.

It was shown in Chap. III that the deformation of a body, if within the elastic limit, is proportional to the applied force. The diagram shows that the maximum internal stresses occur at the outer fiber AH and EL, for the deformation is greatest at these points, and decreases to zero at the center CD or JK.

Axes CD and JK are called the *neutral axes of the sections* or *axes of no stress.* It is proved in mechanics that the neutral axis of a section of a beam passes through the center of gravity of the section.

In addition to the bending of a beam, there is a tendency of one portion of a beam to slip past the adjacent portion. This tendency is called *shear*, and the resistance offered by the fibers of the beam is called the *shearing stress.* Shearing stresses are parallel to the plane of the section. In order that the student may get a better idea of internal shearing and bending stresses, Fig. 306 is drawn to show the fibers on each side of a section.

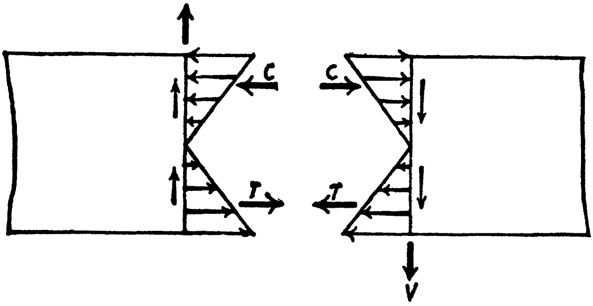


FIG. 306.

The vertical arrows are the resisting shearing forces. Since they are parallel, they are combined into a single force V. Notice that for a V upward on one plane, there is and equal and opposite V on the other plane.

The fiber stresses resisting bending above the neutral axis are compression forces, and those below the center are tension forces. Since they are parallel forces, they may be combined into *single resultants C and T at points two-thirds the distance from the center to the top and bottom of the beam, respectively* (see Art. 117, Center of Gravity of an Area, with respect to the triangle).

Now the two parts of the beam are each in equilibrium under the action of internal stresses and the external loads and forces. Then we may write the three equations of condition, *viz.*,

$$\Sigma X = 0, \qquad \Sigma Y = 0, \qquad \Sigma M = 0$$

1. *The sum of the vertical loads and forces to the left of a section of a beam is equal to the sum of the vertical internal stresses across the section or is equal to the shear V.*

2. *The sum of the horizontal external loads and forces is equal to the sum of the horizontal internal stresses. But all the external forces are vertical.*

Then

$$\Sigma X = T - C = 0, \qquad \text{or } T = C$$

Then T and C form a couple, called the "resisting couple" or "internal-stress couple."

3. *The moment of all the external loads and forces to the left of a section of a beam, about the neutral axis of that section, is equal to the moment of the internal stress couple.*

The three preceding statements are equally true for the right-hand section. But, as a matter of custom, all problems will be solved for simple beams, by dealing with the portion of a beam to the left of a section when possible.

131. Shear and Shear Diagrams.—In order that we may know how the shear changes in going from one end of the beam to the other, problems will be solved in which the shear is calculated at sections 1 ft. apart, beginning with the left end. In general, shear is represented by V. At a point 1 ft. from the left end, represent it by V_1, at 2 ft. by V_2, etc. At the section just over the right edge of the left support, represent it by V_0; at a concentrated load, just to the left of the point, by V_r'; and to the right V_r''.

Example 1.—Find the shear for a simple beam 10 ft. long carrying a concentrated load of 1,000 lb.

Solution.—Each reaction is 500 lb. (Neglect the weight of the beam.)

Then $V_0 = 500$; $V_1 = 500$; $V_2 = 500$; $V_3 = 500$; $V_4 = 500$; $V_5' = 500$; $V_5'' = 500 - 1{,}000 = -500$; $V_6 = V_7 = V_8 = V_9 = V_{10} = -500$. Draw a line AB, 10 ft. long, below the beam of Fig. 307. At A, lay off $AC = 500$ lb. by scale. At 1, draw $1D = 500$ lb. $= V_1$.

At 2, $2E = 500$ lb. $= V_2$, etc. At F, draw $KF = 500 - V_5'$ and $KF' = -500 = V_5''$, etc., to B. Join C, D, E, F, and F' to H. This figure is called the *shear diagram.*

The vertical lines, or ordinates, drawn from the axis AB to the points on CF and $F'H$ give the values of the shearing stress at the corresponding sections of the beam. This diagram shows that at the supports the shear has the maximum value and also that, at K, it passes through zero.

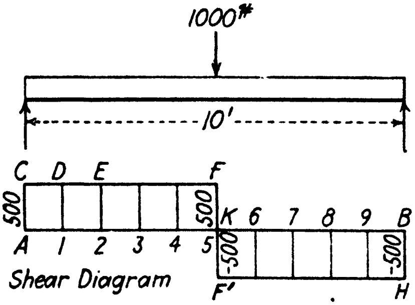


FIG. 307.

Example 2.—Find the shear diagram for a beam 10 ft. long, carrying a uniformly distributed load of 100 lb. per lin. ft., including its own weight (Fig. 308).

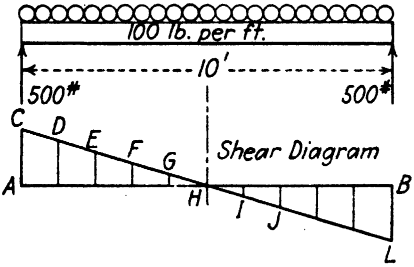


FIG. 308.

Solution.—Each reaction is one-half the load of 1,000 lb., or 500 lb. Then $V_0 = 500$ lb. Between the 1-ft. section and the left end, there are 100 lb. of load; for the 2-ft. section, 200 lb. of load; etc.

Then $V_1 = 500 - 100 = 400$; $V_2 = 500 - 200 = 300$; $V_3 = 500 - 300 = 200$; $V_4 = 500 - 400 = 100$; $V_5 = 500 - 500 = 0$; $V_6 = 500 - 600 = -100$; $V_7 = 500 - 700 = -200$; $V_{10} = 500 - 1{,}000 = -500$.

Take AB as an axis 10 ft. long, and plot the values of V as in the preceding problem, obtaining the points C, D, E, F, G, H, I, J, and L.

Since the ordinates decrease by equal amounts, the points are on a straight line CL. Thus, the shear diagram is obtained. It shows that the maximum shear occurs at the supports and zero shear at the center.

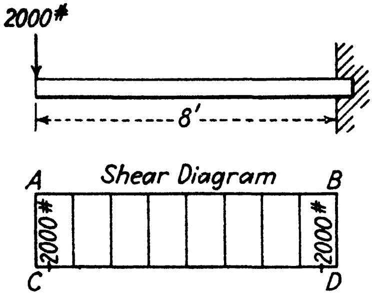


FIG. 309.

The student should notice that, in the first two examples, the value of the shear at the right support is equal and opposite to the reaction at that support. This fact is a check on the accuracy of the calculation.

Example 3.—Construct the shear diagram for a cantilever beam fixed at one end. The beam is 8 ft. long and carries a load of 2,000 lb. on the free end (Fig. 309).

Solution.—There is but one support, *viz.*, the wall, and the reaction is 2,000 lb. The diagram for centilevers is best drawn from the free end.

Then $V_0 = -2{,}000$; $V_1 = -2{,}000$; $V_2 = V_3 = V_4 = V_5 = V_6 = V_7 = V_8' = -2{,}000$ lb. $V_8 = 0$.

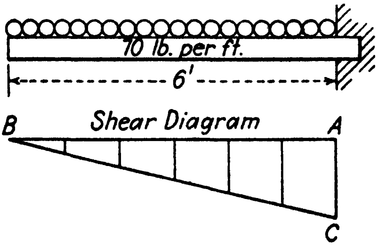


FIG. 310.

The values of V are all equal. Therefore, the shear diagram is the straight line CD parallel to AB.

Example 4.—Construct the shear diagram for a cantilever 6 ft. long, carrying a uniformly distributed load of 70 lb. per ft. of length (Fig. 310).

Solution.—The total load is 420 lb. Then the right-hand reaction is 420 lb.

Then $V_0 = 0$; $V_1 = -70$; $V_2 = -140$; $V_3 = -210$; $V_4 = -280$; $V_5 = -350$; $V_6' = -420$.

Plotting the values of V as in the preceding problems, we obtain CB as the shear diagram.

Example 5.—Construct the shear diagram for a simple beam 12 ft. long, carrying a uniform load of 100 lb. per ft., a concentrated load of 5,000 lb. 4 ft. from the left end, and 8,000 lb. 10 ft. from the left end (Fig. 311). Also find the distance x from the left end where the shear passes through zero.

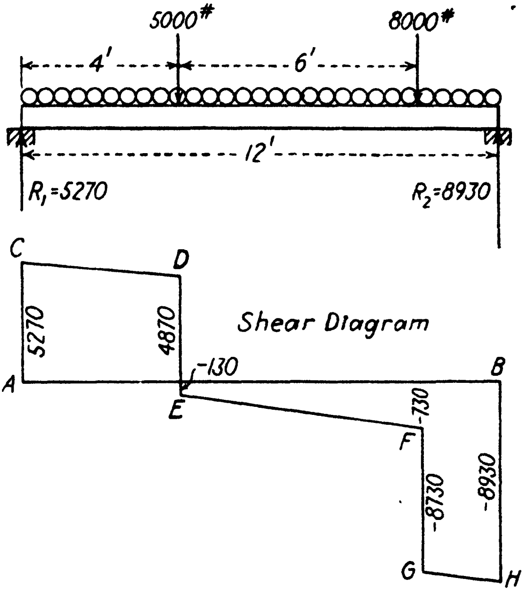


FIG. 311.

Solution.—The reactions must be figured first. Take moments about the left end, remembering that the 100 × 12 = 1,200 lb. of distributed load may be placed at the center of the beam for moments.

$$12R_2 = 5{,}000 \times 4 + 1{,}200 \times 6 + 8{,}000 \times 10$$

$$R_2 = 8.930 \text{ lb.}$$

$$R_1 = 5{,}000 + 1{,}200 + 8{,}000 - 8{,}930 = 5{,}270 \text{ lb.}$$

Now

$$\begin{aligned}
V_0 &= 5{,}270\\
V_1 &= 5{,}270 - 100 = 5{,}170\\
V_2 &= 5{,}270 - 200 = 5{,}070\\
V_3 &= 5{,}270 - 300 = 4{,}970\\
V_4' &= 5{,}270 - 400 = 4{,}870\\
V_4'' &= 4{,}870 - 5{,}000 = -130\\
V_{10}' &= -130 - 600 = -730\\
V_{10}'' &= -730 - 8{,}000 = -8{,}730\\
V_{12} &= -8{,}730 - 200 = -8{,}930
\end{aligned}$$

Plotting the values of V as found, we find $CDEFGH$ to represent the shear diagram.

Since the shear diagram crosses AB at E, the shear passes through zero at $x = 4$.

132. Bending-moment Diagram.—In Art. 130, it was stated that the moment of the internal-fiber-stress couple at any section of a beam is equal to the moment of the external loads or forces to the left of the section. In order that the designer may select a beam to carry a given load, he must know how the bending moment changes at sections from one end of the beam to the other. For this purpose, the bending moment is calculated at sections 1 ft. apart and is plotted just as in shear. The figure thus obtained is called the *bending-moment diagram.*

As in the shear diagram, bending moments will be represented by M with a subscript.

Sign of Bending Moments.—In Chap. V, the moment of a force was taken positive if rotation were counterclockwise. In bending moments, it is the direction of rotation of the resisting bending-stress couple with which we are concerned. Figure 306 shows that when the external forces tend to turn clockwise the resisting moment is counterclockwise, or positive. Therefore, in dealing with the left-hand portion of a beam, if the external forces tend to rotate clockwise they are taken positive and those rotating counterclockwise are taken negative.

In Chap. XIV on Deflection, Typical Cases, another very workable rule for determining the sign of the bending moment will be given. However, the bending moment in a cantilever or in the overhanging part of a beam is always taken as negative.

Example 1.—A beam 10 ft. long carries a load of 1,000 lb. at its center (Fig. 312). Construct the moment diagram.

Solution.—Each reaction is 500 lb. At the left end, $M_0 = 0$. At the 1-ft. section, the only force to the left of the section is 500 lb., with a moment arm 1 ft. Then

$$M_1 = 500 \times 1 = 500 \text{ ft.-lb.}$$
$$M_2 = 500 \times 2 = 1{,}000 \text{ ft.-lb.}$$

$$M_3 = 500 \times 3 = 1{,}500 \text{ ft.-lb.}$$
$$M_4 = 500 \times 4 = 2{,}000 \text{ ft.-lb.}$$
$$M_5 = 500 \times 5 = 2{,}500 \text{ ft.-lb.}$$

Now, at section 6 ft. out, there is 500 lb. upward with moment arm 6 ft. and 1,000 lb. downward with arm 1 ft.

$$\therefore M_6 = 500 \times 6 - 1{,}000 \times 1 = 2{,}000 \text{ ft.-lb.}$$

Then

$$M_7 = 500 \times 7 - 1{,}000 \times 2 = 1{,}500 \text{ ft.-lb.}$$
$$M_8 = 500 \times 8 - 1{,}000 \times 3 = 1{,}000 \text{ ft.-lb.}$$
$$M_9 = 500 \times 9 - 1{,}000 \times 4 = 500 \text{ ft.-lb.}$$
$$M_{10} = 500 \times 10 - 1{,}000 \times 5 = 0$$

Now, draw a line AB, 10 ft. long as in Fig. 312. Since $M_0 = 0$, the point is at A. At 1, erect $1H$ perpendicular to AB and 500 ft.-lb. in length to a convenient scale. Then $M_1 = 1H$. Similarly, draw $2K = M_2 = 1{,}000$; $3E = M_3 = 1{,}500$; $4F = M_4 = 2{,}000$; $5G = M_5 = 2{,}500$; $6H = M_6 = 2{,}000$; etc., to $M_{10} = 0 =$ point B. Since these values of M change by equal amounts for equal lengths of AB, the points A, H, K, E, F, and G are on the straight line AG and the remaining points on GB. The moment diagram then consists of the straight lines AG and GB. The advantage of such a diagram is that the ordinate for any point on the diagram, or line is the bending moment for the corresponding section of the beam.

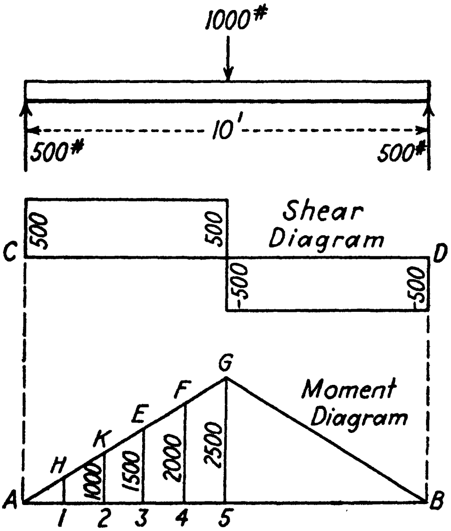


FIG. 312.

The shear diagram has been constructed in Fig. 312. It shows the maximum bending moment at the point where the shear passes through zero. This relation is a perfectly general one for any beam. In order that it may be verified, both shear and bending moments will be constructed for every problem.

Example 2.—A beam 12 ft. long carries a uniform load of 100 lb. per foot of length, including its own weight (Fig. 313). Find the maximum bending moment.

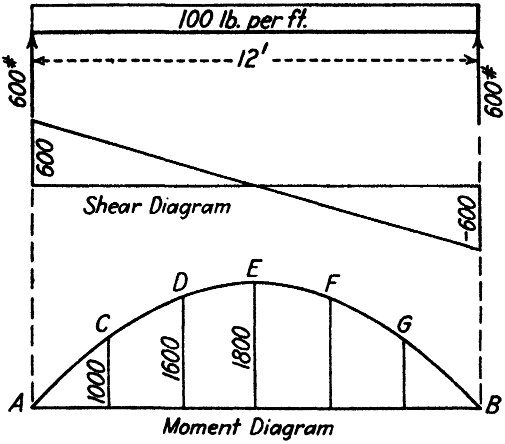


Fig. 313.

Solution.—The total load is 12 × 100 = 1,200 lb. Each reaction is 600 lb. Then $M_0 = 0$.

To the left of the 1-ft. section there are a force of 600 lb. up and a uniform load of 100 lb. down. This load may be considered as concentrated at the center of the 1-ft. section for moments. Its moment arm is ½.

Then

$$
\begin{aligned}
M_1 &= 600 \times 1 - 100 \times \tfrac{1}{2} = 550 \\
M_2 &= 600 \times 2 - 200 \times 1 = 1{,}000 \\
M_3 &= 600 \times 3 - 300 \times 1\tfrac{1}{2} = 1{,}350 \\
M_4 &= 600 \times 4 - 400 \times 2 = 1{,}600 \\
M_5 &= 600 \times 5 - 500 \times 2\tfrac{1}{2} = 1{,}750 \\
M_6 &= 600 \times 6 - 600 \times 3 = 1{,}800 \\
M_7 &= 600 \times 7 - 700 \times 3\tfrac{1}{2} = 1{,}750
\end{aligned}
$$

The student will find the values for M_8, M_9, M_{10}, etc., are just the same as M_5, M_4, M_3, M_2, etc., respectively, owing to the symmetrical loading.

By plotting the values of M as in the previous problem, the points A, C, D, E, F, G, H, etc., are obtained. The curve drawn through them is called the *moment diagram.*

The maximum moment is $M_6 = 1{,}800$ ft.-lb. at the center of the beam. The figure for the shear diagram shows that the shear passes through zero for the same point, which verifies the theorem that the bending moment is a maximum where the shear passes through zero.

Example 3.—A beam 10 ft. long carries a uniform load of 200 lb. per ft. of length and a concentrated load of 2,000 lb. 4 ft. from the left end (Fig. 314). Construct the shear and bending-moment diagrams.

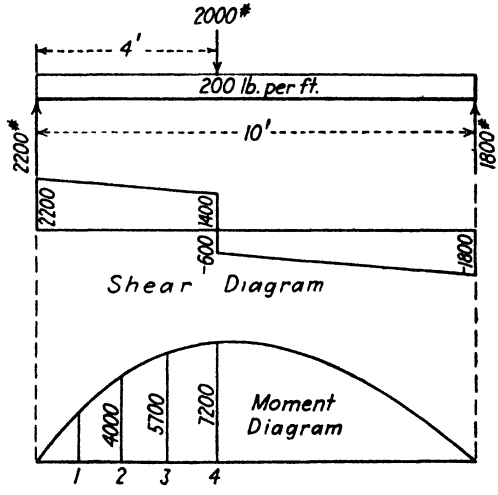


FIG. 314.

Solution.—Taking moments around the left end,

$$10R_2 = 2{,}000 \times 4 + 2{,}000 \times 5$$
$$R_2 = 1{,}800 \text{ lb.}$$
$$\therefore R_1 = 4{,}000 - 1{,}800 = 2{,}200 \text{ lb.}$$

Now $V_0 = 2{,}200$, $V_4' = 1{,}400$

$$V_4'' = 1{,}400 - 2{,}000 = -600$$
$$V_{10} = -600 - 1{,}200 = -1{,}800$$
$$M_0 = 0$$
$$M_1 = 2{,}200 \times 1 - 200 \times \tfrac{1}{2} = 2{,}100$$
$$M_2 = 2{,}200 \times 2 - 400 \times 1 = 4{,}000$$
$$M_3 = 2{,}200 \times 3 - 600 \times 1\tfrac{1}{2} = 5{,}700$$
$$M_4 = 2{,}200 \times 4 - 800 \times 2 = 7{,}200$$

$$
\begin{aligned}
M_5 &= 2{,}200 \times 5 - 1{,}000 \times 2\tfrac{1}{2} - 2{,}000 \times 1 = 6{,}500\\
M_6 &= 2{,}200 \times 6 - 1{,}200 \times 3 - 2{,}000 \times 2 = 5{,}600\\
M_7 &= 2{,}200 \times 7 - 1{,}400 \times 3\tfrac{1}{2} - 2{,}000 \times 3 = 4{,}500\\
M_8 &= 2{,}200 \times 8 - 1{,}600 \times 4 - 2{,}000 \times 4 = 3{,}200\\
M_{10} &= 2{,}200 \times 10 - 2{,}000 \times 5 - 2{,}000 \times 5 = 0
\end{aligned}
$$

The maximum moment is 7,200 ft.-lb. at the distance 4 from the left end, where the shear passes through zero.

Example 4.—A cantilever 6 ft. long carries a load of 1,500 lb. at its end. Construct the shear and moment diagrams (Fig. 315).

Solution.—The reaction $R_1 = 1{,}500$.

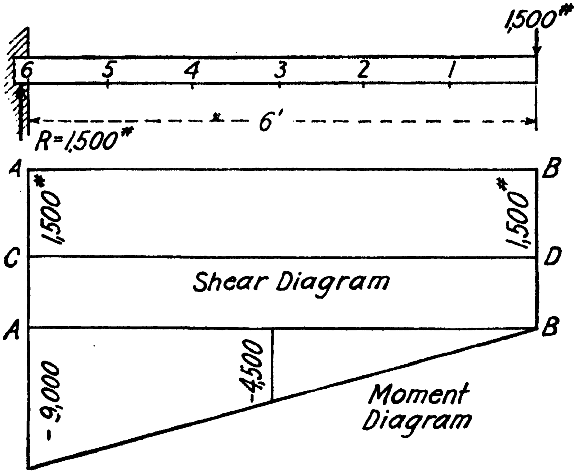


FIG. 315.

In the case of a cantilever beam, the origin is best taken at the free end, because of the downward reaction of the wall itself on the portion of the beam built into the wall. By taking the free end as the origin and figuring the moments for the portion of the beam between the section and the free end, the undetermined downward reaction may be eliminated.

$$
\begin{aligned}
M_0 &= 0\\
M_1 &= -1{,}500 \times 1 = -1{,}500\\
M_2 &= -1{,}500 \times 2 = -3{,}000\\
M_6 &= -1{,}500 \times 6 = -9{,}000
\end{aligned}
$$

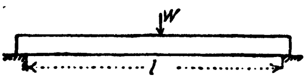
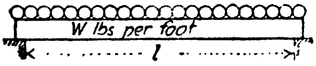
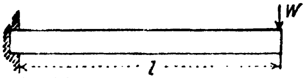
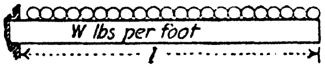
The maximum moment is $-9{,}000$ at the wall.

In this case, it is evident that the shear passes through zero at the edge of the wall, where the maximum bending moment is found. However, if the downward pressure of the wall on the built-in portion of the beam were known and the shear diagram constructed for that, the shear would pass through zero at the outer edge of the wall.

For a cantilever beam, the maximum bending moment and shear are at the wall.

133. Maximum Bending Moments for Various Types of Beam.—In many cases simple beams carry uniformly distributed loads or loads concentrated at the center. Also, cantilevers carry uniform loads or loads at the end. For each of these cases, the maximum bending moments always occur at definite points and have definite values. The preceding table will be found useful. The student should verify each formula.

TABLE VIII

	Maximum bending moment in foot-pounds	Maximum shear at support in pounds	Maximum deflection
 Simple Beam. Central Load	$\frac{1}{4}Wl$	$\frac{W}{2}$	$-\frac{Wl^3}{48EI}$
 Simple Beam. Uniform Load	$\frac{1}{8}Wl$	$\frac{W}{2}$	$-\frac{5}{384}\frac{Wl^3}{EI}$
 Cantilever Beam. Load on End	Wl	W	$-\frac{Wl^3}{3EI}$
 Cantilever Beam. Uniform Load	$\frac{Wl}{2}$	W	$-\frac{Wl^3}{8EI}$

134. Bending Moment from the Area under the Shear Diagram.—It is beyond the scope of this work to prove the relation between shear and bending moment. But the relation may be verified in one or two simple cases. Take, for example, a simple beam (Fig. 316), with a load *W* in the center. Construct the shear diagram. Choose any point *A* at the distance *y* from the left end *C*.

The bending moment at A is

$$\frac{W}{2} \times y = \frac{Wy}{2}$$

The area of the rectangle $DAEF$ is

$$\frac{W}{2} \times y = \frac{Wy}{2}$$

$\therefore$ B.M. = area under the shear curve to the left of the section.

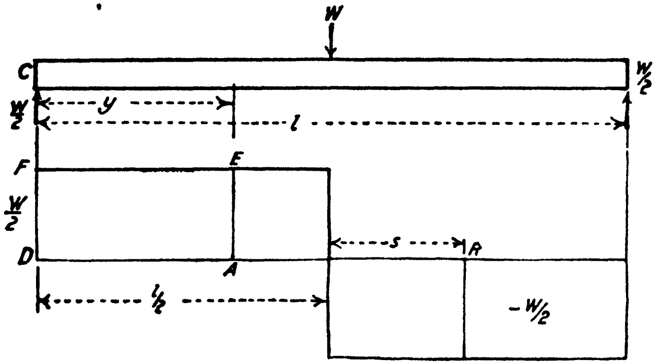


FIG. 316.

Area to the left of the center is

$$\frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$$

which is the bending moment.

At R, s units to the right of the center, the area is equal to that above minus the area below the axis, or

$$\frac{W}{2} \times \frac{l}{2} - \frac{W}{2} \times s = \frac{Wl}{4} - \frac{Ws}{2}$$

The bending moment is

$$\frac{W}{2}\left(\frac{l}{2} + s\right) - Ws = \frac{Wl}{4} + \frac{Ws}{2} - Ws = \frac{Wl}{4} - \frac{Ws}{2}$$

which verifies the relation.

The theorem is as follows:

The bending moment at any section of a beam resting on supports is equal to the area of the shear diagram to the left of the section, plus the bending moment at the left support.

If M_x is the moment at any section, S_x the area of the shear diagram to the left of the section, and M_o the moment at the left support,

$$M_x = S_x + M_o \tag{43}$$

In all simple beams except those with an overhang, $M_o = 0$. Then $M_x = S_x$.

In Fig. 312, the moment at 5 ft. is the area of the shear diagram to the left of 5. The altitude of the rectangle is 500 and the base 5.

Then

$$S_5 = 500 \times 5 = 2{,}500$$
$$\therefore M_5 = S_5 = 2{,}500$$

which is the result obtained in the moment diagram.

Example 1.—Construct the moment diagram of Fig. 313 by means of the shear diagram.

The principle just stated is especially useful when the loading is not symmetrical. The shear diagram should first be constructed so as to find where it passes through zero. The area of the shear diagram to the left of this point is the maximum bending moment, when $M_o = 0$.

Example 2.—A beam of 15-ft. span carries a uniform load of 400 lb. per lin. ft., a concentrated load of 5,000 lb. 5 ft. from the left and 2,000 lb. 4 ft. from the right end (Fig. 317). Find the maximum bending moment.

Solution.—The reactions must first be found. Taking moments about the left end,

$$\begin{aligned} 15R_2 &= 400 \times 15 \times \tfrac{15}{2} + 5{,}000 \times 5 + 2{,}000 \times 11 \\ R_2 &= 6{,}133 \text{ lb.} \\ R_1 &= 6{,}000 + 2{,}000 + 5{,}000 - 6{,}133 = 6{,}867 \text{ lb.} \end{aligned}$$

The shear diagram is *BCEFGH*. The shear passes through zero at the point *D*. The maximum bending moment is then at the section of the beam under the 5,000-lb. load. Its value is the area *ABCD.*

$$\text{Area } ABCD = \frac{6,867 + 4,867}{2} \times 5 = 29,335$$

$$\therefore \text{B.M.} = 29,335 \text{ ft.-lb.}$$

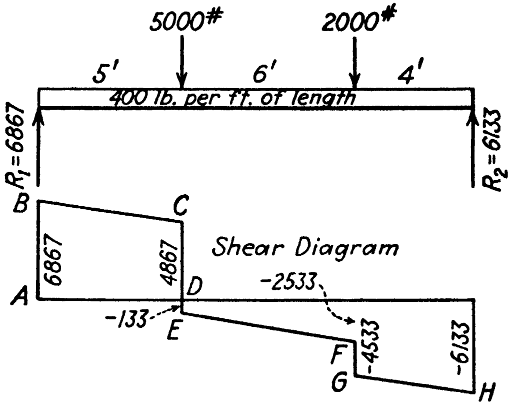


FIG. 317.

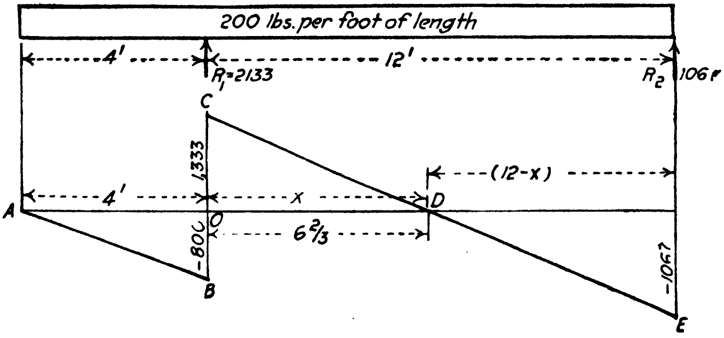


FIG. 318.

Example 3.—A beam 16 ft. long rests on supports, one 4 ft. from the left end and one at the right end, and is loaded as shown in Fig. 318.

Solution.—To find the reactions, take moments around the right end. The total load of 3,200 lb. may be taken at a point 8 ft. from R_2. Then

$$12R_1 = 3,200 \times 8$$

or

$$R_1 = 2{,}133 \text{ lb.}$$
$$R_2 = 1{,}067$$

At the left end of the beam the shear is 0, and it becomes less by 200 lb. for every foot of length, or -800 to the left side of support R_1. It then is increased by 2,133.

$$OC = 2{,}133 - 800 = 1{,}333 \text{ lb.}$$

At E, the shear is $-1{,}067$, which is just equal to the reaction R_2 as it should be.

The diagram crosses the axis at O and D. The bending moment must be figured for both points.

The area to the left of O is

$$-\frac{800 \times 4}{2} = -1{,}600 \text{ ft.-lb.}$$
$$\therefore \text{B.M. at } O = 1{,}600 \text{ ft.-lb.}$$

To find D, use the proportion

$$\frac{x}{12 - x} = \frac{1{,}333}{1{,}067}$$
$$x = 6\tfrac{2}{3} \text{ ft.}$$

The total area to the left of D is

$$OCD - ABO = \frac{1{,}333 \times 6\frac{2}{3}}{2} - 1{,}600$$
$$= 4{,}444 - 1{,}600 = 2{,}844.$$
$$\therefore \text{B.M. at } D = 2{,}844 \text{ ft.-lb.}$$

Since the bending moment at D is greater than at O, numerically the maximum bending moment for the beam is 2,844 ft.-lb.

In applying the principle of areas to a cantilever beam, the student should notice that the bending moment at any section is the area under the shear diagram between the section and the free end of the beam.

Example 4.—Find the maximum bending moments for each of the examples in Art. 132, by means of the area of the shear diagram.

Example 5.—Draw the shear and bending-moment diagrams for each of the problems at the end of this chapter.

135. Maximum Bending Moment in a Beam Due to a Load Increasing Uniformly to the Center of the Beam.—

In Fig. 319 is shown a beam carrying a brick wall so constructed that $HB = AH$, *i.e.*, $\angle HAB = 45°$. The wall is of uniform thickness throughout, of t ft., and weighs w lb.

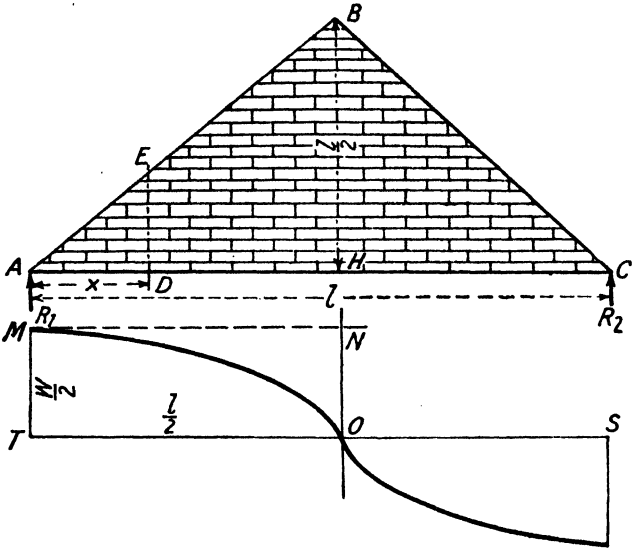


Fig. 319.

per cu. ft. The weight of the section ADE = volume × $w = \frac{1}{2}AD \times DE \times t \times w = \frac{1}{2}x^2tw$. Then the weight of

$$AHB = \frac{1}{2}\left(\frac{l}{2}\right)^2 tw = \frac{l^2}{8}tw$$

$$W = \frac{l^2}{4}tw$$

Notice that t and l are in feet.

Each reaction is $W/2$. Then the shear at any point D is the reaction at A minus the load between A and D, or

$$V = \frac{W}{2} - \frac{1}{2}x^2tw$$

Now, with TS as the x axis and T as the origin, plot the shear curve. When $x = 0$, $V = W/2$; when $x = l/2$, $V = 0$.

Since the load is symmetrical, for points equidistant from the center of the beam, the values of V will be equal but opposite in sign. Plotting for $x = l/4$, $x = 3l/8$, etc., we obtain a curve as shown in the figure. Each half is a parabola. The shear passes through zero at O, giving the point of maximum bending moment. $M =$ area TMO. From higher mathematics, the area under a segment of a parabola such as TMO is two-thirds the area of the circumscribed rectangle $TMNO$. But

$$TMNO = \frac{l}{2} \times \frac{W}{2} = \frac{Wl}{4}$$

$$\therefore M = \frac{2}{3} \times \frac{lW}{4} = \frac{Wl}{6} \text{ ft.-lb.} \qquad (44)$$

If angle $HAB = \Theta$, then $HB = (l/2) \tan \Theta$, and

$$W = \frac{1}{2}l \times \frac{l}{2} \tan \Theta \times tw$$

$$= \frac{l^2}{4} tw \tan \Theta$$

the expression for M remaining the same.

Example.—Let $t = 9$ in., $w = 150$ lb. per cu. ft., $l = 20$ ft., and $\angle HAB = \angle\Theta = 45°$. To find M, the solution is

$$W = \tfrac{1}{4} \times (20)^2 \times \tfrac{9}{12} \times 150 = 11{,}250 \text{ lb.}$$
$$M = \tfrac{1}{6} \times 20 \times 11{,}250 = 37{,}500 \text{ ft.-lb.}$$
$$= 450{,}000 \text{ in.-lb.}$$

Since the loading shown in Fig. 319 may be considered a standard type, the bending moment may be obtained from the general equation $M = Wl/6$. For this case, there is no need for drawing the shear diagram. Since the shear must be equal to zero at the middle of the span, owing to the symmetry of the load, the maximum moment will occur at this point. Then the moment at $l/2$ is equal to the moment of the reaction minus the moment of one-half of the triangular load, or

$$M = \frac{Wl}{2} - \frac{W}{2} \times \frac{1}{3} \times \frac{l}{2}$$

$$M = \frac{Wl}{4} - \frac{Wl}{12} = \frac{3Wl}{12} - \frac{Wl}{12} = \frac{Wl}{6} \qquad (44a)$$

The moment arm of the triangle AHB is the distance from the center of gravity of the triangle to the middle of the beam, or $\frac{1}{3} \times l/2$.

136. Moving Loads.—Hitherto, in calculating the shear and bending moments the assumption has been made that the loads on the beam were in a fixed position. There are many cases where this is not true, however; a loaded truck or a train passing over a bridge or a piano being moved over a floor are examples of loads maintaining a fixed distance from each other. It is evident that the bending moment under any load will not necessarily be the same as that under another load; nor will the moment be constant under any load as it moves from one end of the beam to the other. The problem, then, is to find the maximum bending moment under each load separately. The greatest of all these values is the one used in designing the beam.

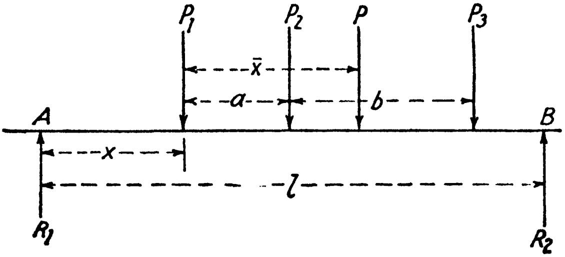


FIG. 320.

In discussing the theory of this problem, only three loads will be considered. The conclusions drawn will be general, however, and can be applied to any number of loads. Let P be the resultant of the loads. First, determine reaction R_1 (Fig. 320).

$$\Sigma M_B = -lR_1 + (l - x - \bar{x})P = 0$$

$$R_1 = \frac{(l - x - \bar{x})P}{l}$$

For bending moment under P_1, consider the section of the beam to the left of P_1.

$$\text{B.M.} = xR_1 = \frac{(lx - x^2 - \bar{x}x)P}{l}$$

$$= -[x^2 - (l - \bar{x})x]\frac{P}{l}$$

But bending moment is a maximum when the expression in the brackets is greatest, since P is constant. Now, in algebra, it is shown that the expression $ax^2 + bx + c$ is greatest or least when

$$x = -\frac{b}{2a}$$

In $x^2 - (l - \bar{x})x$, $a = 1$, $b = -(l - \bar{x})$.
Then $x = (l - \bar{x})/2$ gives a maximum or a minimum. But the minimum moment at the supports being zero, then the preceding value of x is for the maximum moment. It is now evident that the distance from A to the resultant P is $(l + \bar{x})/2$. That is, the center of the beam is midway between the load P_1 and the resultant P of all the loads.

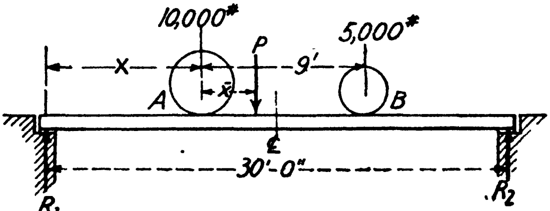


FIG. 321.

The rule may be stated as follows: *The maximum bending moment under any load of a set of moving loads occurs when that load is as far on one side from the center of the beam as the resultant of all the loads on the beam is on the other side.*

Example 1.—Two loads of 10,000 and 5,000 lb., 9 ft. apart, roll over a beam 30 ft. long (Fig. 321). Find the maximum bending moment.

Solution.—First find the position of the resultant with reference to A, by taking moments about A.

$$15{,}000\bar{x} = 9 \times 5{,}000$$
$$\bar{x} = 3 \text{ ft.}$$

Then

$$x = \frac{l - \bar{x}}{2} = \frac{30 - 3}{2} = 13.5 \text{ ft.}$$

or 10,000 is 1.5 ft. to the left of the center of the beam, and P is 1.5 ft. to the right.

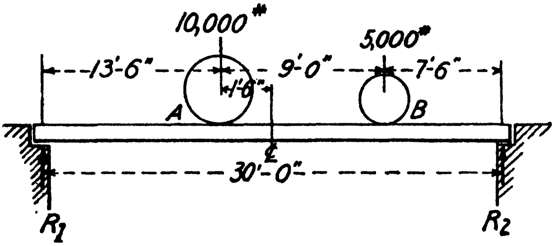


FIG. 322.

Figure 322 shows the position of the loads.

$$\Sigma M_{R_2} = -30R_1 + 16.5 \times 10{,}000 + 7.5 \times 5{,}000 = 0$$
$$R_1 = 6{,}750 \text{ lb.}$$

Now, cut the beam under the 10,000-lb. load, and take moments about this section of the loads on the left, or

$$\Sigma M_A = 13.5 \times 6{,}750 = 91{,}125 \text{ ft.-lb.}$$

which is the maximum moment that is developed under the 10,000-lb. load. Since P is 6 ft. from B, the maximum moment under the 5,000-lb. load occurs when P and B are each 3 ft. from the center of the beam as shown in Fig. 323.

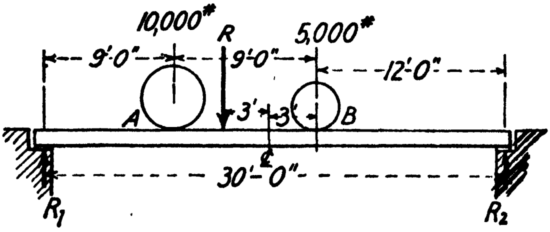


FIG. 323.

$$\Sigma M_{R_2} = -30R_1 + 10{,}000 \times 21 + 5{,}000 \times 12$$
$$R_1 = 9{,}000 \text{ lb.}$$

Next, take a section of the beam at B, and find moments of the load to the left of B.

$$\Sigma M_B = 18 \times 9{,}000 - 9 \times 10{,}000 = 72{,}000 \text{ ft.-lb.}$$

This is the maximum moment developed under the 5,000-lb. load.

Since the maximum moment under A is greater than that under B, 91,125 ft.-lb. is the maximum for the beam.

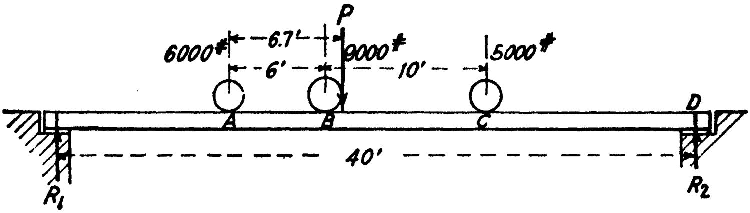


FIG. 324.

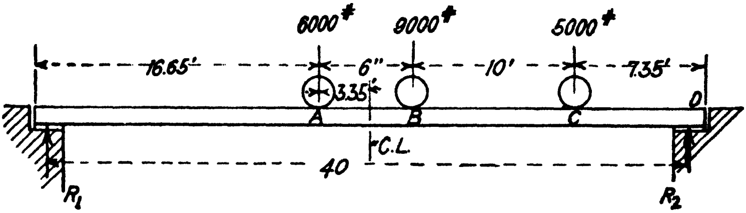


FIG. 325.

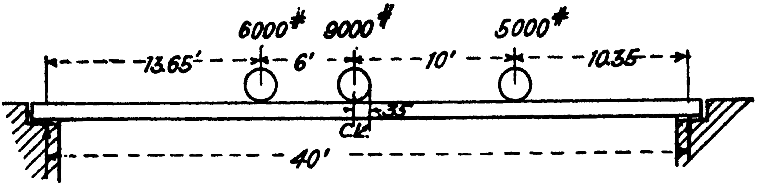


FIG. 326.

Example 2.—Three loads of 6,000, 9,000, and 5,000 lb. are placed 6 and 10 ft. apart, respectively (Fig. 324). They are moved across a bridge of span 40 ft. Find the maximum bending moment.

Solution.—First locate the resultant P with reference to point A.

$$20{,}000x = 6 \times 9{,}000 + 16 \times 5{,}000$$
$$x = 6.7 \text{ ft.}$$

The free-body diagrams will be shown for each load in the position for maximum moments and results given. The work of verifying results is left to the student.

$$M_A = 16.65 \times 8{,}325 = 138{,}611 \text{ ft.-lb.}$$
$$M_B = 157{,}061 \text{ ft.-lb.}$$
$$M_C = 117{,}810 \text{ ft.-lb.}$$

$\therefore M = 157{,}061$ ft.-lb. is the maximum bending moment.

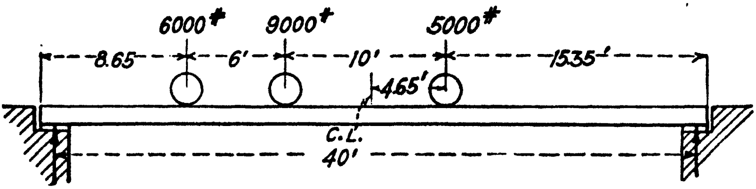


Fig. 327.

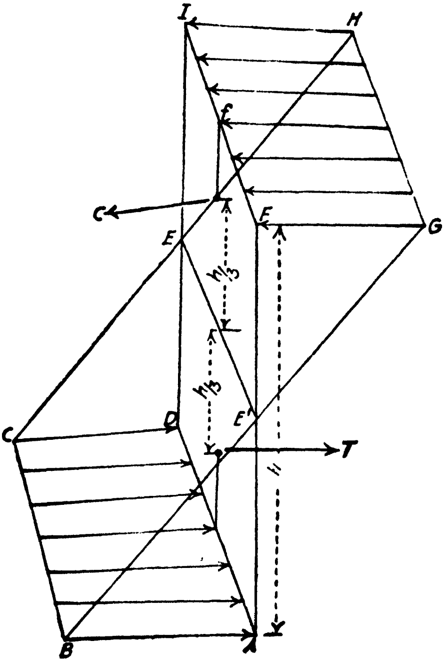


Fig. 328.

137. Resisting Moment, for a Rectangular Section.—In Art. 130, it was shown that the fiber stresses over any cross section of a beam were a maximum at the extreme outer fibers, decreasing to zero at the neutral axis. These stresses are represented in Fig. 328 for any section.

Let *ADIF* be the rectangular cross section of the beam. The maximum compressive stresses along *IF* are $IH = p$ lb. per sq. in., and the maximum tensile stresses along *AD* are *BA*, equal to $IH = p$ lb. per sq. in. of surface. The total compressive stress on *E'EIF* is the volume of the wedge *E'EIHGF*. But the volume of the wedge is the area of the base *E'FG*, times the thickness b.

$$E'FG = \frac{1}{2}p \times \frac{h}{2} = \frac{ph}{4},$$

$$\therefore V = \frac{bhp}{4}.$$

But this is the sum of all the parallel compressive stresses, which has been called C. Then

$$C = \frac{bhp}{4} \text{ lb.}$$

The center of the triangle *E'FG* is two-thirds of the distance from the vertex E'. Then C may be placed at a point

$$\frac{2}{3} \times \frac{h}{2} = \frac{h}{3}$$

above the neutral axis *E'F*.

The volume of the wedge *ABCDEE'* representing the tensile forces is equal to the volume of the wedge *E'EIHGF*. If T stands for the resultant tensile stresses, then

$$T = \frac{bhp}{4}$$

placed at a point $h/3$ below *E'E*.

Now the moment of a couple is one force times the distance between the forces. Therefore, the moment of the resisting couple is

$$\frac{pbh}{4} \times \frac{2h}{3} = \frac{pbh^2}{6}$$

Then from Art. 130,

$$\text{B.M.} = \frac{pbh^2}{6} \tag{45}$$

The bending moment must be in inch-pounds. By means of equation 45 the greatest fiber stress at any section may be found. But the designer is usually concerned with the section at which the bending moment is a maximum for the beam. The value of p at this section is the maximum fiber stress for the beam.

Table XVII gives safe values of p to be used in the design of beams of the various materials. If the load, span, and size of a beam are given, the developed fiber stress p may be found. This value should not exceed the safe values given in the table. Or if the span, size of beam, and safe fiber stress are given, the greatest load that the beam may safely carry may be found by Eq. (45).

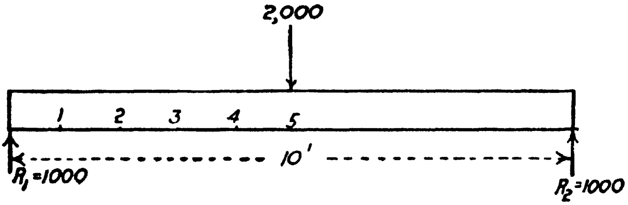


FIG. 329.

Example 1.—A simple beam of southern pine 6 by 8 in. and 10 ft. long carries a concentrated load of 2,000 lb. at the center (Fig. 329). Find the greatest fiber stresses developed at sections 3 ft., 5 ft., and 8 ft. from the left end. (Neglect weight of beam.)

Solution.—The reactions are 1,000 lb. each. At the 3-ft. section the only external load to the left is 1,000 lb.

$\therefore$ B.M. $= 1{,}000 \times 3 \times 12 = 36{,}000$ in.-lb.

Also, $b = 6$, $h = 8$.

Substituting in Eq. (45), we have

$$36{,}000 = \frac{p \times 6 \times 64}{6}$$

$$p = 560 \text{ lb./sq. in.}$$

At the 5-ft. section,

B.M. $= 5 \times 1{,}000 \times 12 = 60{,}000$ in.-lb.

$$\therefore 60{,}000 = p \times 6 \times \tfrac{64}{6}$$

$$p = \frac{60{,}000}{64} = 933 \text{ lb./sq. in.}$$

At the 8-ft. section,

B.M. $= 8 \times 1{,}000 \times 12 - 2{,}000 \times 3 \times 12 = 24{,}000$ in.-lb.

$$\therefore 24{,}000 = p\frac{\cancel{6} \times 64}{\cancel{6}}, \qquad p = \frac{24{,}000}{64} = 375 \text{ lb./sq. in.}$$

These results verify the statement that the greatest fiber stress is found at the section of maximum bending moment, *viz.*, the center of the beam in this case.

Example 2.—A simple wooden beam with 15-ft. span is to carry a uniform load of 400 lb. per lin. ft., including its own weight. The timber is 6 in. wide. Find the depth that will carry the load safely.

Solution.—The maximum bending moment is at the center and from Table VIII, Art. 133, is equal to $\frac{1}{8}Wl$.

But

$$W = 15 \times 400, \text{ and } l = 15 \times 12 = 180 \text{ in.}$$
$$\therefore \text{B.M.} = \tfrac{1}{8} \times 15 \times 400 \times 15 \times 12 = 135{,}000 \text{ in.-lb.}$$

A safe value of $p = 1{,}000$ lb. $b = 6$ in.

Then

$$\frac{1{,}000 \times \cancel{6} \times h^2}{\cancel{6}} = 135{,}000, \qquad h^2 = 135, \qquad h = 11.6 \text{ in.}$$

The nearest commercial height is 12 in. Therefore, a beam 6 by 12 in. should be used.

Example 3.—Find the dimensions of a square timber used as a cantilever beam 8 ft. long to carry a load of 1,500 lb. at the free end.

Solution.—The maximum bending moment that occurs at the free end is Wl from Table VIII, Art. 133.

$$l = 8 \times 12 = 96 \text{ in.,} \qquad \text{and } W = 1{,}500 \text{ lb.}$$

since the section is square,

$$p = 1{,}000 \text{ lb.,} \qquad \text{and } b = h$$

Then, from Eq. (45),

$$1{,}500 \times 96 = \frac{1{,}000 \times b^3}{6}$$
$$b^3 = 864, \qquad b = 9.5$$

The nearest commercial size is 10 by 10 in.

Example 4.—A floor that carries a load of 100 lb. per sq. ft. of floor area including the weight of the floor is supported by wooden beams of 12-ft. span, spaced 24 in. from center to center. If the beams are 4 in. wide, find their depth.

Solution.—Each beam must support a floor area of $12 \times 2 = 24$ sq. ft. Then

$$W = 24 \times 100 = 2{,}400 \text{ lb.}$$
$$l = 12 \times 12 = 144 \text{ in.,} \qquad p = 1{,}000 \text{ lb.,} \qquad \text{and } b = 4 \text{ in.}$$

Since the load is uniformly distributed, the maximum bending moment is $\frac{1}{8}Wl$ from Table VIII, Art. 133. Substituting in Eq. (45), we have

$$\frac{1}{\cancel{8}} \times \overset{18}{\cancel{144}} \times 2{,}4\cancel{0}\cancel{0} = \frac{1{,}0\cancel{0}\cancel{0} \times 4 \times h^2}{6}$$

$$h^2 = 65, \qquad h = 8+ \text{ in.}$$

The nearest commercial size larger is 10 in. Therefore, use a 4- by 10 in. beam. Since a 4- by 10-in. beam is considerably larger than necessary, this beam will care for its own weight.

Example 5.—A balcony projecting 6 ft. from a wall is supported by beams 6 by 10 in. spaced 3 ft. apart. Find the uniformly distributed load per square foot of floor area that the floor will carry, using a safe fiber stress of 800 lb. per sq. in.

Solution.—Each beam acting as a cantilever must carry $6 \times 3 = 18$ sq. ft. of floor load. If w is the load per square foot, then

$$W = 18w \text{ lb.}$$

Maximum bending moment is $\frac{1}{2}Wl$ by Table VIII, Art. 133.

$$l = 6 \times 12 = 72 \text{ in.}, \qquad p = 800, \qquad b = 6, \qquad \text{and } h = 10$$

Substituting in Eq. (45),

$$\frac{1}{2} \times 18w \times 72 = \frac{800 \times 6 \times 10 \times 10}{6}$$

$$\therefore w = 124.6 \text{ or } 125 \text{ lb./sq. ft.}$$

The 125 lb. must include the weight of floor and beam. The allowable live load is the remainder.

The weight per cubic foot of the types of lumber most commonly used varies from 35 to 50 lb. An average value of 40 lb. would be close enough for all practical purposes. For more accurate values, consult a handbook.

138. Live Load and Moisture Effects.—All the safe working stresses of Table XVII (Appendix) are based on the assumption that the load is applied quietly, or is a dead load.

Now, it is proved in mechanics that when a load is suddenly applied to a structure the tensile and compressive stresses set up are twice as great as if the load were applied gradually. Thus, in designing buildings, bridges, etc., the engineer must study carefully the types of load to be carried by the structure before deciding on a safe working stress.

In the first place, he must decide on the dead and live loads. A dead load is usually thought of as the weights of the beams, struts, floor boards, and such pieces of furniture and machinery as may be built in with the structure. On a roof truss a snow load is considered a dead load. A live load is one applied suddenly, as a train rushing onto a bridge, a heavy gust of wind against a building, people dancing on a floor, crowds in a grandstand, etc. All such live loads set up violent vibrations and, as stated before, cause strains in the structural members from one to two times as great as those caused by dead loads. It is usually considered that people walking around on a floor will cause stresses and 1½ times as great as a dead load of equal weight. For example, an average fiber stress in southern pine is 1,000 lb. per sq. in. for a dead load. Then, for a floor of a church, hall, or school building where dancing is not allowed, a safe working stress would be 750 lb., whereas if dancing were permitted or heavy machinery were to be moved about, the safe fiber stress would be 500 lb.

139. Nominal and Real Sizes of Timber.—The values for sizes to meet the requirements of the American Lumber Standards can be found in tabular form in the 1927 *Proceedings* of the American Railway Engineering Association, pages 325 to 326, 341 to 344, and 361. In these tables a nominal 3 by 12 timber has real dimensions of 2⅝ by 11¼. The nominal cross-sectional area is 36 sq. in., and the real area is 29.9 sq. in. The real area is 83 per cent of the nominal. In the same way, a 10 by 10 has real dimensions 9½ by 9½. The real area is 92.3 per cent of the nominal. These results show that the smaller the cross-section of the timber the greater the difference between the nominal and real areas. In using large timbers, little attention need be paid to the real dimensions, for a commercial size larger than the calculated nominal size is chosen. This factor alone will usually take care of the difference between the nominal and real dimensions. In finding the load that a timber of small cross section will carry, allowance should be made for its real cross section.

In example 5, Art. 137, the load found was 125 lb. per sq. ft. of floor area. A 6- by 10-in. timber has real dimensions of 5½ by 9½ in., or a cross-sectional area 87 per cent of the nominal. Then the true allowable floor load is

$$0.87 \times 125 = 109 \text{ lb.}$$

In Example 2, Art. 137, $h = 11.6$ was the calculated depth, and a 6- by 12-in. beam chosen. The real depth of this beam is 11¼ and for certain types of loads might not be considered large enough. In example 3, Art. 137, a 9.5- by 9.5-in. timber was found and a 10- by 10-in. timber selected, which is correct.

The designer should have a copy of Lumber Association tables to check computations and make corrections when necessary.

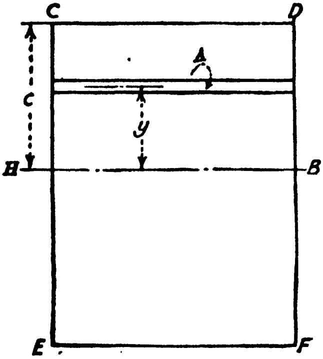


FIG. 330.

140. Section Modulus.—In Art. 137, the equation for the resisting bending moment was derived for the special case of a wooden beam. The same formula will now be derived in such terms that it will be applicable to the cross section of the beams of any shape. For simplicity, a rectangular section will be used. The result will be true in general, however, as proved in more advanced works on strength of materials.

Let Fig. 330 be the section of a rectangular beam. HB is the neutral axis. Take a small rectangle of area A, at the distance y from HB. Let s be the stress per square inch on A. Then the total stress S on A is

$$S = sA$$

Taking moment of S about HB, we have

$$\text{Moment } S = ysA \qquad (m)$$

But the fiber stress is 0 at HB and increases to a maximum value p at CD, the extreme fiber. Then

$$s:y::p:c$$

or

$$s = \frac{py}{c}$$

Substituting in (m) above, we have

$$\text{Moment } S = y \cdot \frac{py}{c}A = \frac{p}{c}y^2A$$

To find the bending moment of all the fibers across CF, we must take a very great number of rectangles such as A. Since these moments are referred to a common axis, they may be added. Then

$$\text{Total moment} = \Sigma\frac{p}{c}y^2A$$

But p/C is the same for every term of the summation

$$\Sigma\frac{p}{c}y^2A = \frac{p}{c}\Sigma y^2A$$

From Art. 124, $\Sigma y^2A = I$ the moment of inertia about HB. Since the moment of the resisting couple is equal to the bending moment of the external loads to the left of the section,

$$\mathbf{B.M.} = \mathbf{p}\frac{\mathbf{I}}{\mathbf{c}} \qquad (46)$$

where p is the maximum fiber stress developed at the section, and c the distance from the neutral axis to the extreme fiber.

The expression I/c is called the *section modulus.* Equation (46) may then be written

$$\text{B.M.} = p \times \text{section modulus}$$

If the base *EF* of Fig. 330 is b and the height h,

$$I = \frac{1}{12}bh^3, \qquad \text{and } c = \frac{h}{2}$$

Substituting in Eq. (46),

$$\text{B.M.} = \frac{p\frac{1}{12}bh^3}{h/2} = \frac{pbh^2}{6}$$

which is Eq. (45).

Handbooks of the various steel companies give tables of details for I beams, angle irons, channels, etc. By means of Eq. (46), if the load on a beam is given the bending moment may be computed. The section modulus is then easily found. The handbook will give the necessary size and weight of beam to carry the load. Or, if the size of the beam is given, the section modulus is found from the tables. The safe load is then easily calculated.

For the convenience of the student, tables will be found in the Appendix containing the properties of some of the standard shapes and sizes of beams.

141. Strength of a Beam.—The strength of a beam of given length depends on the load that it will carry. The greater the load, the stronger the beam. Since p is a safe working fiber stress, its value is determined by the kind of material used. The bending moment for a beam of given length varies with the load.

Then bending moment varies with the section modulus. The greater the section modulus, the greater the load and the strength of the beam.

If a beam if 6 by 10 in. and placed with long dimension vertical,

$$\text{B.M.} = p \times \frac{6 \times 10^2}{6} = 100p$$

If the short dimension is placed vertical,

$$\text{B.M.} = p \times \frac{10 \times 6^2}{6} = 60p$$

But B.M. in Fig. 331 is $\frac{1}{4}W_1l$ and in Fig. 332 is $\frac{1}{4}W_2l$,

$$\therefore \frac{\frac{1}{4}W_1l}{\frac{1}{4}W_2l} = \frac{100p}{60p} = 1\frac{2}{3}$$

$$\therefore W_1 = 1\frac{2}{3}W_2$$

That is, the beam will carry a load $\frac{5}{3}$ greater in the first case than in the second.

The strength of beams of the same length and cross section will depend on the kind of loading and supports.

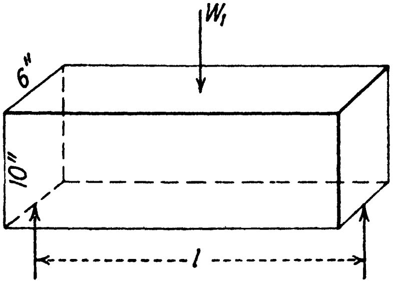


FIG. 331.

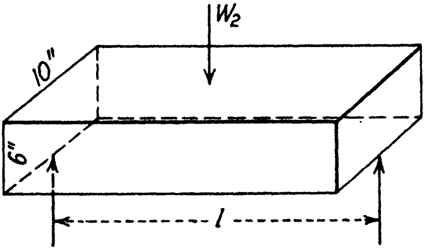


FIG. 332.

By comparing the bending moments in Table VIII, Art. 133, the following conclusions may be drawn:

1. *A cantilever beam with a uniformly distributed load is twice as strong as when the load is at the free end.*

2. *A simple beam with a central load is twice as strong as a cantilever with a uniform load and four times as strong as one with a load on the free end.*

3. *A simple beam with a uniformly distributed load is twice as strong as a simple beam with a central load, four times*

as strong as a cantilever with a uniform load, and eight times as strong as a cantilever with a load on the free end.

For example, if a simple beam 6 by 10 in. and 12 ft. long carries a uniform load of 16,000 lb., it would carry only 8,000 central load, 4,000 as cantilever uniformly loaded, or 2,000 as a load on the free end.

142. Illustrations.—Some examples will now be given to show the application of Eq. (46).

Example 1.—Find the required size and weight of an I beam 16 ft. long, resting on end supports to carry 40,000 lb. uniformly distributed.

Solution.—For structural steel, $p = 16{,}000$ lb. From Table VIII, Art. 133, B.M. $= Wl/8$. From Eq. (46),

$$\frac{1}{\cancel{8}} \times \overset{5}{\cancel{40{,}000}} \times \cancel{16} \times 12 = \cancel{16{,}000} \times \text{section modulus}$$

$$\text{Section modulus} = 60 \text{ cu. in.}$$

In Table XI, Appendix the column headed S.M. contains the section modulus for the corresponding beams in the first column. The nearest value greater than 60 is 60.5, corresponding to a 15-in. 45-lb. I beam.

If the 40,000 lb. is to include the weight of the beam, then the actual load carried is

$$40{,}000 - 16 \times 45 = 39{,}280 \text{ lb.}$$

If the load must be 40,000 lb., the beam should now be figured on the basis of 40,720 lb. as the load, and what fiber stress p is developed should be determined.

$$\tfrac{1}{8} \times 40{,}720 \times 16 \times 12 = p \times 60.8$$

$$p = \frac{24 \times 40{,}720}{60.5} = 16{,}150 \text{ lb./sq. in.}$$

Since this value exceeds the safe stress of 16,000 lb., the next larger size should be chosen, *viz.*, 15 in. 50 lb. with section modulus = 64.2.

Example 2.—Find the central load that an 8-in. 25.5-lb. I beam will carry when the span is 12 ft.

Solution.—Let W = the load and $p = 16{,}000$ lb./sq. in.

The beam weighs $12 \times 25.5 = 306$ lb. This is a uniformly distributed load. A central load of 153 lb. will produce the same fiber stress. Then the total load to use is $(W + 153)$ lb. Substituting in Eq. (46),

$$\tfrac{1}{4}(W + 153) \times 12 \times 12 = 16{,}000 \times 17.1 \quad \text{(from Table XI, Appendix)}$$

$$W + 153 = \frac{16{,}000 \times 17.1}{36} = 7{,}600 \text{ lb.}$$

$$\therefore W = 7{,}447 \text{ lb. approximately}$$

Example 3.—Select the proper I beams to carry a uniform load of 50,000 lb., with the following spans: 8, 10, 12, and 14 ft.

Example 4.—A 12-in. 50-lb. I beam with a span of 20 ft. is to carry a uniform load of 450 lb. per ft. and a concentrated load of 6,000 lb. 5 ft. from the left end. Find the maximum fiber stress developed.

The total uniform load is 450 + 50 = 500 lb. per ft. First, construct the shear diagram, to find where it passes through zero. This point will be found at the section 7 ft. from the left end. Maximum bending moment occurs at this point.

$$\text{B.M.} = 9{,}500 \times 7 - 3{,}500 \times \tfrac{7}{2} - 6{,}000 \times 2 = 42{,}250 \text{ ft.-lb.}$$

From a handbook, section modulus is 50.6. Then

$$42{,}250 \times 12 = p \times 50.6$$

$$p = \frac{507{,}000}{50.6} = 10{,}000 \text{ lb./sq. in.}$$

This value is well within safe limits.

Example 5.—Design a simple southern pine timber beam to carry a uniform load of 10,000 lb. on a 10-ft. span.

Solution.—A safe fiber stress to use is 1,000 lb. per sq. in. The danger section is in the middle. From Eq. (46)

$$\frac{1}{8} \times 10{,}000 \times 10 \times 12 = 1{,}000 \times \frac{bh^2}{6}$$

$$\therefore bh^2 = 900$$

Any number of cross sections could be chosen. For example, $b = 4$ and $h = 16$ would give $4 \times 256 = 1{,}024$. But this beam would require lateral bracing.

If $b = 8$ and $h = 12$, then $bh^2 = 8 \times 144 = 1{,}152$.

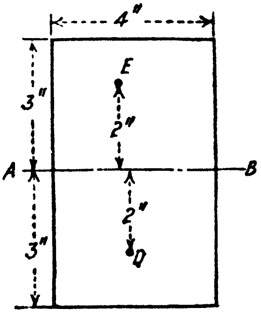


FIG. 333.

But an 8- by 12-in. beam is really 7½ by 11½. On this basis, $bh^2 = 7\tfrac{1}{2} \times 11\tfrac{1}{2} \times 11\tfrac{1}{2} = 992$. Then an 8- by 12-in. beam will carry the load.

Example 6.—A 4- by 6-in. southern pine beam with a 10-ft. span carries a uniform load of 1,600 lb. The cross section of the beam is shown in Fig. 333. Find the resultant compressive and tensile stresses at mid-section of the beam and also at a section 2 ft. from the left support.

Solution.

a. The resultant compressive stress is a force C at a point E, two-thirds of the distance from the neutral axis, or 2 in. above. Also the resultant tensile stress is a force T applied at D, 2 in. below AB. From Art. 130, $C = T$. They form a couple with moment $4C$.

But B.M. = $4C$. Then $\frac{1}{8} \times 1{,}600 \times 10 \times 12 = 4C$.

$$C = T = 6{,}000 \text{ lb.}$$

b. At a section 2 ft. from the left support, B.M. $= 800 \times 2 - 320 \times 1 = 1{,}280$ ft.-lb. Then

$$1{,}280 \times 12 = 4C$$
$$C = 3{,}840 \text{ lb.}$$

This principle is of importance when beams are to be spliced, for the value of C or T is the load that the splice plate must carry from one part of the beam to the other.

143. Bending in Oblique Planes.—Most of the beams used in actual practice have at least one axis of symmetry and more generally two. The beams are placed with one of these axes vertical, and the load is applied to the beam in the vertical plane containing this axis. All of the foregoing problems have been of this type. There are numerous cases where the line of action of the load does not coincide with a principal axis. Thus the purlin in a roof truss supports a load that may be defined as unsymmetric. The true neutral axis makes an angle with the principal axes that is not easily determined. But this problem can be solved quite easily by obtaining the components of the load parallel to the two principal axes and then computing the bending moments and stresses separately.

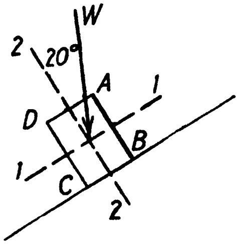


Fig. 334.

Thus, let the line of action of the load make an angle of 20° with the axis 2-2 of the beam. The component of the load parallel to the axis 2-2 is $W \cos 20°$, and the component parallel to the axis 1-1 is $W \sin 20°$. These components are shown in Fig. 334. $W \cos 20°$ will produce a bending about the axis 1-1, and $W \sin 20°$ causes a bending about the axis 2-2. The stresses produced by these two components are parallel to the long axis of the beam and may be added algebraically.

Example 1.—A 4- by 6-in. wooden cantilever beam 6 ft. long supports an end load of 100 lb. (Fig. 335). The 6-in. dimension of the cross section makes an angle of 20° with the vertical. What will be the stresses?

Solution.—First resolve the load into components parallel to the two principal axes. Then 100 cos 20° = 100 × 0.9397 = 93.97 lb.

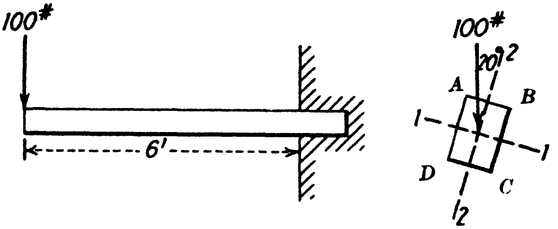


Fig. 335.

is the force in the plane 2-2. And 100 sin 20° = 100 × 0.342 = 34.2 lb. is the force in the plane 1-1. Then

$$M_{1\text{-}1} = 93.97 \times 6 \times 12 = 6{,}750 \text{ in.-lb.}$$

$$p = \frac{Mc}{I} = \frac{6{,}750 \times 3}{(4 \times 6)/12} = 281.4 \text{ lb./sq. in.}$$

This value will be the amount of the tensile stress at corners *A* and *B* and compressive stress at *C* and *D*.

$$M_{2\text{-}2} = 34.2 \times 6 \times 12 = 2{,}462.4 \text{ in.-lb.}$$

$$p = \frac{2{,}462.4 \times 2}{(6 \times 4^3)/12} = 153.9 \text{ lb./sq. in.}$$

This value will be the amount of the tensile stress at the corners at *A* and *D* and compressive stress at *B* and *C*.

To find the resultant stresses, it is best to obtain the summation for each corner. If we use [−] to denote tension and [+] to denote compression, then at *A* there is tension due to both components of the load, and the values for stress are as follows:

At *A*,

$$S_A = -281.4 - 153.9 = -435.3 \text{ lb./sq. in. tension}$$

At *B*,

$$S_B = -281.4 + 153.9 = -127.5 \text{ lb./sq. in. tension}$$

At C,

$$S_C = +281.4 + 153.9 = +435.3 \text{ lb./sq. in. compression}$$

At D,

$$S_D = +281.4 - 153.9 = +127.5 \text{ lb./sq. in. compression}$$

Thus, somewhere between B and C, the stress will be zero, and this point will be on the neutral axis. Similarly, some point between A and D will have zero stress, and this point is on the neutral axis which is then determined.

Since the stresses are proportional to the distances from the neutral axis, then if we let x be the distance from the neutral axis to C,

$$\frac{x}{435.3} = \frac{6 - x}{127.5}$$

Solving, we have

$$127.5x = 435.3 - 435.3x$$
$$562.8x = 435.36 = 2{,}611.8$$
$$x = \frac{2{,}611.8}{562.8} = 4.64 \text{ in.}$$

Similarly, another point on the neutral axis will be found 4.64 in. from corner A.

If the exact location of the true neutral axis is wanted, it can be easily located with reference to the principal axes from the data just obtained.

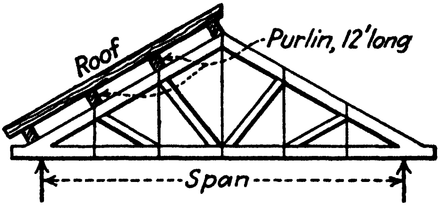


FIG. 336.

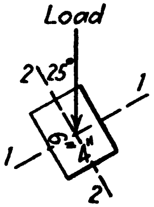


FIG. 336a.

Example 2.—A roof purlin 4 by 6 in. and 12 ft. between supports carries a distributed load of 1,200 lb. (Fig. 336). The 6-in. side of the beam makes an angle of 25° with the vertical. The line of action of the load is through the center of gravity of the beam. What are the stresses at the corners of the beam?

144. Simple Beam with Load Increasing Uniformly from One End.—This type of loading is found in a bin that becomes partly filled when the coal or grain which is poured in from one side assumes its angle of repose so that the floor is just covered. The force due to the pressure of the water against a vertical lock gate varies in this manner.

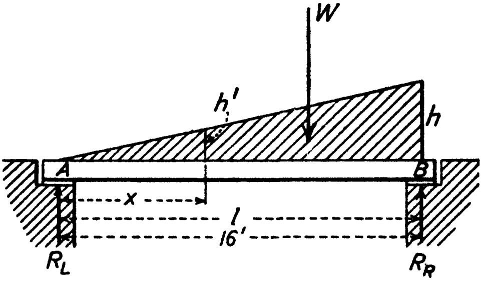


FIG. 337.

In deriving the general expression for moment, the reactions will first be determined. Find R_L by taking moments about B.

$$R_L \cdot l = \frac{Wl}{3} \qquad \text{or } R_L = \frac{W}{3}$$

$$W = \frac{lthw}{2}, \qquad \text{and } R_L = \frac{lthw}{2} \times \frac{1}{3} = \frac{lthw}{6}$$

Also,

$$W_x = \frac{xth'w}{2}, \qquad \text{but } \frac{h'}{h} = \frac{x}{l} \text{ and } h' = \frac{hx}{l}$$

Then

$$W_x = \frac{xthxw}{2l} = \frac{x^2thw}{2l} = \text{load to any point } x$$

Shear at any point,

$$V_x = R_L - W_x, \qquad V_x = \frac{lthw}{2} - \frac{x^2thw}{2l}$$

$M_{\max}$ occurs where $V = 0$.
Then

$$\frac{lthw}{2} - \frac{x^2thw}{2l} = 0$$

Cancelling, we have

$$x = \frac{l}{\sqrt{3}}$$

Moment

$$M_x = R_L x - \frac{W_x x}{3} \qquad \text{or } M_x = \frac{lthwx}{6} - \frac{x^2thw}{2l}\,\frac{x}{3}$$

Then at $x = \frac{l}{\sqrt{3}}$,

$$M = \frac{lthwl}{6\sqrt{3}} - \frac{l^2thwl}{3l \times 6\sqrt{3}}$$

$$M = \frac{lthwl}{6\sqrt{3}} - \frac{lthwl}{18\sqrt{3}}$$

Factoring the value of $W = lthw/2$, we have

$$M = \frac{2Wl}{9\sqrt{3}} \tag{47}$$

Problems

240. A wooden beam 12 ft. long is supported at the ends. If the actual cross section is 6 by 8 in. and the allowable bending stress 1,100 lb. per sq. in., what total uniform load can the beam support?

241. In Prob. 240, if the beam weighs 12 lb. per ft. of length, what equal concentrated loads placed at the quarter points of the span can it support?

242. In Prob. 240, if the allowable bending stress is 1,200 lb. per sq. in. and the actual cross section of the beam is 6 by 9 in., what may be the span if the total uniform load is 6,000 lb?

243. A southern pine beam 8 by 10 in. by 12 ft. long carries its safe load. With this load, what will be the fiber stress 2 in. from the top at a section 6 ft. from the left support? Find the resultant compression and tension at the 5-ft. and 6-ft. sections. Draw the shear diagram, and from it determine the total shear at the 6-ft. section.

244. What should be the size of I beams spaced 4 ft. on centers to carry a balcony load of 225 lb. per sq. ft.? The balcony is constructed as a cantilever and extends 8 ft. beyond the wall.

245. What should be the proper size of an I beam with a 16 ft. span, if it overhangs each support 5 ft.? The beam carries 7,000 lb. at the middle, 8,000 lb. at the left end, and 4,000 lb. at the right end.

246. Design a southern-pine beam 16 ft. long to carry a uniform load of 200 lb. per ft., a concentrated load of 1,200 lb. 6 ft. from the right end, and 800 lb. 4 ft. from the right end. What is the resultant compression at a section 6 ft. from the left end?

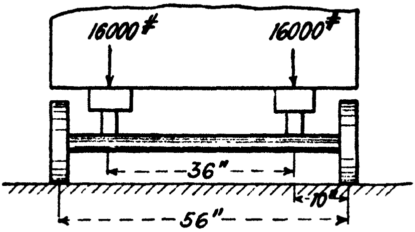


FIG. 338.

247. What should be the diameter of the hollow axle for the truck shown in Fig. 338? The safe unit bending stress is 10,000 lb. per sq. in. The outside diameter is to be twice the inside diameter.

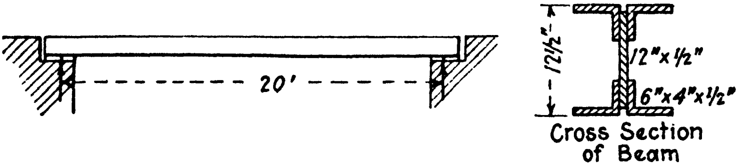


FIG. 339.

248. What total uniformly distributed load and an equal concentrated load at the middle could the beam of Fig. 339 support? What is the stress at the quarter point?

249. A 6- by 10-in. southern pine beam is 12 ft. long. What is the ratio of the total uniform loads it can support if the 6-in. and the 10-in. dimensions are respectively taken as vertical?

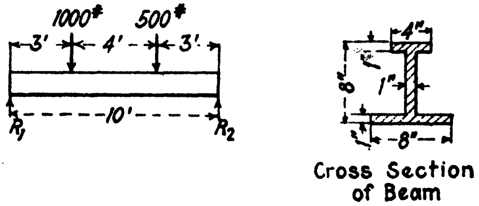


FIG. 340.

250. What are the stresses in the cast-iron beam shown in Fig. 340? Are they safe?

251. What total uniformly distributed load would the beam of Fig. 340 carry?

252. What uniformly distributed load will a 15-in. 45-lb. I beam safely carry on spans of 12 ft. and 16 ft., respectively? What equal concentrated loads at the third points will it carry on those spans?

253. A steel pin in a pin-connected truss is to be considered as a simple beam of 8 in. span. It resists a force of 8,000 lb. at its center. What should be the diameter of this hard-steel pin if the allowable stresses are, respectively, 8,000 lb. for shear, and 16,000 lb. for bending?

254. The cross section of the beam in Fig. 341 is made up by bolting together three 2- by 10-in. planks. What uniform load will the beam support if the allowable bending stress is taken as 1,200 lb. per sq. in.? What is the resultant compression at the center of the span?

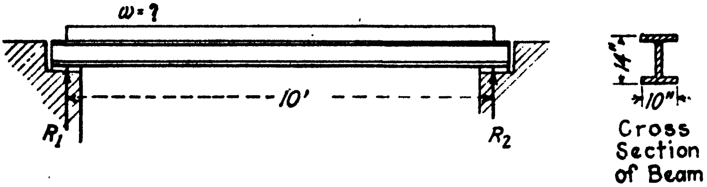


Fig. 341.

255. How far apart may 2- by 12-in. wooden floor joists be spaced if the span is 16 ft.? The live load on the floor is 80 lb. per sq. ft. and the dead load may be taken as 40 lb. per sq. ft. The allowable bending stress is 1,200 lb. per sq. in.

256. In Prob. 255, what will be the allowable load per square foot if the joists are spaced 2 ft. on centers?

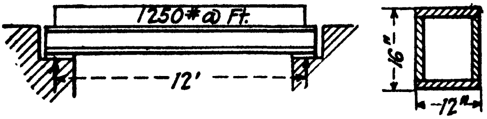


Fig. 342.

257. What will be the maximum stress in the box girder of Fig. 342? The girder is made up by bolting together four 2- by 12-in. planks as shown. The span is 12 ft., and the uniformly distributed load is 1,250 lb. per lin. ft.

258. What must be the sizes of the beams and girders in the interior bay of the steel floor system shown in Fig. 343? The total dead and live load to be supported amounts to 250 lb. per sq. ft.

259. A total uniform load of 16,000 lb. is carried by a beam of Douglas fir. If the span is 14 ft. and the width of beam is 4 in., what is the depth?

260. A floor is supported by 12 in. 35-lb. I beams spaced 8 ft. center to center. What uniformly distributed floor load will they support with a span of 18 ft.?

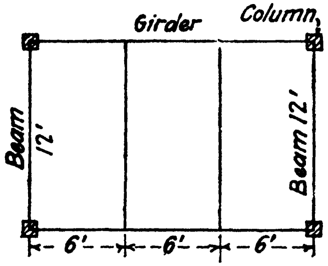


Fig. 343.

261. With the floor load found in Prob. 260, what would be the spacing for a span of 20 ft.?

262. What would be the safe span for the beams of Prob. 260 if the floor load is 220 lb. per sq. ft.?

263. What should be the depth of a western red-cedar beam 4 in. wide and 12 ft. long to carry equal concentrated loads at the third points?

264. What is the ratio of the loads to cause the same fiber stresses in a simple beam, if one load is uniformly distributed and the other is concentrated at the middle?

265. The floor joists in a building are spaced 30 in. apart. What must be the size of southern-pine joists 14 ft. long to carry a total floor load of 80 lb. per sq. ft. and a ceiling load of 40 lb. per sq. ft.?

266. How far apart may the supports for a 10-in. 30-lb. I beam be placed if it supports a load of 8,000 lb. at the middle of the span? The bending stress is limited to 12,000 lb. per sq. in.

267. What must be the size of an oak beam 16 ft. long to permit a load of 5,000 lb. to roll over the floor supported by the beam? It is assumed that the entire load may be supported by a single beam.

268. What is the maximum bending stress in a 15 in. 55-lb. I beam of 24 ft. span if it supports a uniform load of 800 lb., in addition to its own weight, and loads of 6,000 and 9,000 lb., respectively, at 6 ft. and 16 ft. from the right support?

269. Three- by 12-in. southern pine joists are spaced 30 in. apart. The span is 12 ft., and the total dead and live load is 110 lb. per sq. ft. What is the maximum stress due to bending?

270. Select an I beam 24 ft. long, overhanging each support 4 ft., to carry a load of 8,000 lb. at the left end, 6,000 lb. at the center, and 4,000 lb. at the right end. Sketch the shear diagram. What is the moment at each support and at the middle?

271. A balcony supported by cantilever beams 6 ft. long carries 120 lb. per sq. ft. What should be the spacing of 6- by 8-in. white-oak beams.

272. The allowable bending stress in a wooden beam 12 ft. long is 1,000 lb. per sq. in. What will be the value of equal loads placed at the quarter points if the beam is 6 in. wide and 10 in. deep?

273. What uniformly distributed load will an 8- by 8-in. southern pine beam 10 ft. long safely carry if supported at the ends? What load would a similar 4- by 8-in. beam carry? Could two 4- by 8-in. beams fastened together carry the same load as the 8- by 8-in. beam?

274. What safe load will the 12-in. 35-lb. I beam support if it varies from zero at the left end and increases uniformly to a maximum value at the right end (Fig. 337)?

275. What must be the diameter of a steel shaft 10 ft. long between the bearings if the bending stress is 10,000 lb. per sq. in.? A pulley located 3 ft. from one end is subjected to a total pull of 1,600 lb. (Steel weighs 490 lb. per cu. ft.)

276. In a scaffolding used in construction, there is a plank 2- by 12-in. used to support the workmen. The plank is held up by means of brackets 8 ft. apart. If a 200-lb. man stands at the middle and a 150-lb. man stands 1 ft. from the left end, what is the bending stress in the plank?

277. The wall above a show window is 12 in. thick and weighs 120 lb. per cu. ft. (Fig. 344). The clear span is 10 ft., and the distance from center to center of bearing is 11 ft. What must be the size of two channels to carry this wall? ($\angle \theta = 45°$.)

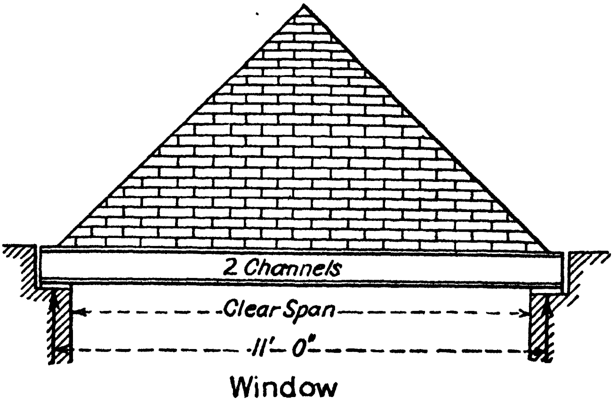


FIG. 344.

278. The brickwork above the display window in a store is supported by two channels. The clear span is 8 ft., and the span of the beam is to be taken as 9 ft. The brickwork is 12 in. thick. What should be the size of the channels? ($\angle \theta = 45°$.)

279. What size of steel beam will be needed to carry the moving loads in Fig. 345? The bending stress is limited to 15,000 lb. per sq. in.

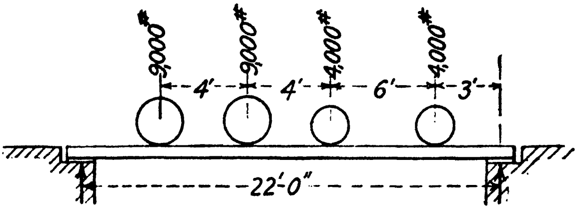


Fig. 345.

280. The maximum pressure between a connecting rod and the crankpin is 120,000 lb. This pressure may be assumed to be uniformly distributed against the pin which is 8 in. long. What must be the diameter of the pin if the allowable shearing stress is 8,000 lb. per sq. in. and the safe bending stress is 12,000 lb. per sq. in.?

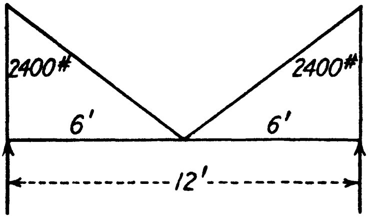


Fig. 346.

281. The 10-in. 25.4-lb. I beam in Fig. 68 has a span of 12 ft. The riveted connections are considered as simple supports. What safe uniform load will this beam carry? How does this value compare with the load found in Prob. 67 (Chap. II)? Explain the difference.

282. What size of I beam will be needed to support the wedge-shaped loads shown in Fig. 346?

CHAPTER XIII

HORIZONTAL AND VERTICAL SHEAR

145. Horizontal Shear.—Shear has been defined as the tendency of two adjacent portions of a body to slide by each other, as the two blades of a pair of shears. The subject of vertical shear in beams has already been considered in Art. 130. There is also a shearing tendency in a plane at right angles to the vertical shearing plane. This is called *horizontal shear.*

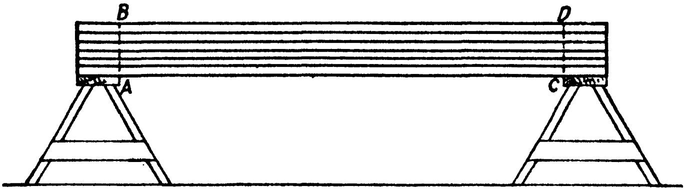


Fig. 347.

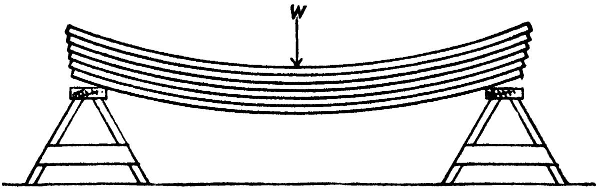


Fig. 347*a*.

The student will get the best idea of horizontal shear from a simple illustration. Place several 12- by 1-in. boards on a couple of supports as shown in Fig. 347. When the boards are of the same length, the ends will be practically straight and even. If, now, a load W is applied at the center of the pile, the boards bend or sag in the middle and the result is as shown in Fig. 347*a*. Each board tends to slide on the one above or below it and in this way moves the ends from their original positions. The horizontal motion of one board over the other is what causes horizontal shear.

Now, if each board is glued securely to the ones next to it and a small load applied, the ends of the pile of boards remain square and even, as they were before the load was applied. There is still a tendency to slide, but the motion is prevented by the glue. This resistance measures the horizontal shear. Again, if the boards are securely bolted together and a load is applied, the boards cannot slip. The shearing resistance is offered by the bolts.

When the pile of boards is replaced by a beam 12 by 5 in. and a load W applied, the beam will bend slightly. There is still the tendency of one horizontal portion to slide past the adjacent face. However, if the beam is not overloaded, the motion is prevented by the internal fibers of the beam, just as it was resisted by the glue or bolts. The resistance that the fibers can offer is called their *horizontal shearing strength.*

Some fibrous materials, such as wood and wrought iron, are more likely to fail from horizontal shear than those materials that have no natural internal cleavage surfaces. Timber has a low shearing strength parallel to the grain, and a short beam fails by horizontal shear before giving way from bending caused by a load. Timbers heavily loaded often have openings or cracks along the sides. These openings are usually considered as a result of seasoning, whereas many of them really show a tendency to horizontal shearing.

146. Unit Horizontal and Vertical Shear at Any Point in a Beam Are Equal.—Figure 348 represents a small cube cut from any part of a loaded beam.

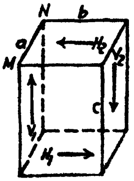


FIG. 348.

Let V_1 and V_2 be the vertical shearing forces and H_1 and H_2 the horizontal shearing forces. The dimension b is very small, so that the effect of the load so far as vertical shear on the two faces is concerned may be neglected and

V_1 may be considered equal to V_2. Similarly,

$$H_1 = H_2$$

Let s_v be the unit vertical shear and s_h the unit horizontal shear. Then

$$V_1 = acs_v, \qquad \text{and } H_1 = abs_h = H_2$$

Since the block is in equilibrium under the shearing forces, moments about the line MN must equal zero, or

$$c \cdot H_1 = b \cdot V_2$$
$$c \cdot abs_h = b \cdot acs_v$$
$$\therefore s_h = s_v \tag{48}$$

In Arts. 14 and 25, it was assumed that the shearing stress was of the same intensity on each square inch of the cross section. In the derivations and the solutions that follow, it will be shown that the unit shearing stress in a beam of symmetrical section is a maximum at the neutral axis and decreases to zero at the outer fibers.

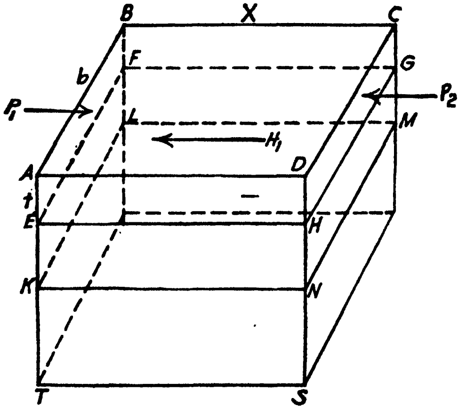


Fig. 349.

The relation between the unit stresses just deduced may be stated thus: *At any point in a member subjected to shearing forces, there exist equal unit shearing stresses in planes mutually at right angles to each other.* The subscripts may be dropped and the equal unit shearing stresses may be designated as s_s.

147. Formula for Horizontal Shear.—Figure 349 represents a section cut from a beam. *ABCD* is the top surface of the beam section, *EFGH* is a horizontal section *t* units below the top, and *KLMN* is the neutral surface of the section.

In Art. 132, it was seen, that in general, bending moment varied from one end of the beam to the other. Let P_1 be the total compressive stress on *ABFE* and P_2 that on *DCGH* caused by the bending. Since P_1 and P_2 are different, they must be balanced by a force H_1 on the surface *EHGF*, and this is the horizontal shear on that surface.

If P_1 is greater than P_2, for equilibrium,

$$H_1 = P_1 - P_2$$

Let f_1 and f_2 be the fiber stresses at the center of gravity of the faces *ABFE* and *CDHG*, respectively, and *A* the area of each of the faces. Then

$$P_1 = f_1A, \qquad \text{and } P_2 = f_2A.$$

But

$$\frac{f_1}{p_1} = \frac{\bar{y}}{c}, \qquad \text{and } \frac{f_2}{p_2} = \frac{\bar{y}}{c}$$

where $\bar{y}$ is the distance from the neutral axis to the center of gravity of *HC*, *c* the distance from the neutral axis to outside fibers, p_1 and p_2 the fiber stresses in the fibers *AB* and *CD*, respectively.

Then

$$H_1 = \frac{p_1\bar{y}A}{c} - \frac{p_2\bar{y}A}{c} = \frac{\bar{y}A}{c}(p_1 - p_2)$$

But $H_1 = s_hbX$ where s_h is the horizontal shearing stress per square inch.

$$\therefore s_h = \frac{\bar{y}A}{Xbc}(p_1 - p_2)$$

If M_1 and M_2 are the bending moments on the right- and left-hand faces, respectively, by Eq. (46),

$$p_1 = \frac{M_1 c}{I}, \qquad \text{and } p_2 = \frac{M_2 c}{I}$$

Then

$$s_h = \frac{\bar{y}Ac}{bcXI}(M_1 - M_2) = \frac{\bar{y}A}{bI}\left(\frac{M_1 - M_2}{X}\right)$$

It is proved in more advanced mechanics that, if X is very small, as in Art. 146,

$$\frac{M_1 - M_2}{X} = V$$

the total vertical shear on either complete beam section BT or SC.

$$\therefore\ s_h = \frac{\bar{y}AV}{Ib} \tag{49}$$

That is, *unit horizontal shear at any point in a beam is equal to the total vertical shear across the vertical section, times the area A above the horizontal section, times the distance from the neutral axis to the center of gravity of the area, divided by the product of the moment of inertia of the cross section of the beam and the width of the beam.*

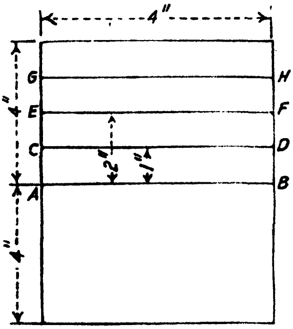


FIG. 350.

Example.—Suppose a simple rectangular beam 4 by 8 in. carries a central load of 1,000 lb. To find the horizontal shearing stress on section 0 in., 1 in., 2 in., 3 in., and 4 in. above the neutral axis at the cross section adjacent to the supports (Fig. 350).

Solution.—In Chap. XII, it was found that maximum shear occurred at the supports and was equal to the reaction at the support. Then

$$V = R = \tfrac{1}{2}W = 500 \text{ lb.}$$

$$I = \frac{1}{12}bh^3 = \frac{4 \times 8 \times 8 \times 8}{12} = \frac{512}{3} = 170.7, \qquad \text{and } b = 4$$

a. At the neutral axis *AB*,

$$A = 4 \times 4 = 16, \qquad \bar{y} = 2$$

$$\therefore\ s_h = \frac{\bar{y}AV}{Ib} = \frac{2 \times 16 \times 500}{170.7 \times 4} = 23.4 \text{ lb.}$$

b. At the section *CD*,

$$A = 4 \times 3 = 12, \qquad \text{and } \bar{y} = 2.5$$

$$s_h = \frac{2.5 \times 12 \times 500}{170.7 \times 4} = 22 \text{ lb. approximately}$$

c. At the section *EF*,

$$A = 4 \times 2 = 8, \qquad \text{and } \bar{y} = 3$$

$$s_h = \frac{3 \times 8 \times 500}{170.7 \times 4} = 17.5 \text{ lb.}$$

d. At the section *GH*,

$$A = 4 \times 1 = 4, \qquad \text{and } \bar{y} = 3.5$$

$$s_h = \frac{4 \times 3.5 \times 500}{170.7 \times 4} = 10.3 \text{ lb.}$$

e. At the top of the section, $s_h = 0$, since $A = 0$.

This problem illustrates the principle that unit $s_h = s_v$ is a maximum at the neutral axis and 0 at the outer fibers.

To study the manner in which the unit shearing stresses vary, the stresses occurring at different distances from the neutral axis may be laid off to scale as shown in Fig. 351. The vectors representing the stresses are of varying length as can be seen from the values computed in the foregoing example. If the ends of the vectors are joined by means of a smooth curve, then any horizontal length between the vertical axis and the curve will represent the unit shearing stress at the corresponding point in the beam. The curve so determined is found to be a parabola.

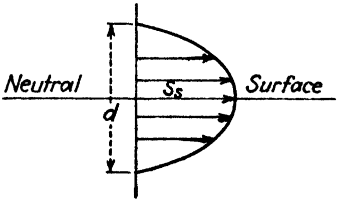


FIG. 351.

If a rectangle b units wide and h in. high is taken, then $A = \frac{1}{2}bh$, and $\bar{y} = h/4$ for the area above the neutral axis. Substituting in Eq. (49),

$$s_h = \frac{(h/4) \times (bh/2) \times V}{\frac{1}{12}bh^3 \times b} = \frac{3}{2}\frac{V}{bh} \qquad (50)$$

But bh is the area of the cross section of the beam and V/bh = average vertical shear.

Therefore, maximum horizontal and vertical shearing stresses are equal to $\frac{3}{2}$ the average vertical shearing stress in a rectangular section.

In designing, the maximum stresses are the ones to be considered. Those for shearing forces occur at the supports. Equation (49) need not be used in a rectangular section. Simply divide shearing force V at the support by the area of the cross section, and multiply the quotient by $\frac{3}{2}$, for maximum $s_h = s_v$.

In the foregoing problem, $V = 500$, and area $= 4 \times 8 = 32$. Then

$$s_h = \tfrac{3}{2} \times \tfrac{500}{32} = 23.4$$

which was obtained by Eq. (49).

Example.—A simple beam 4 by 6 in. carries a uniform load of 2,000 lb. Find the maximum s_h.

Solution.—$V = R_1 = \frac{1}{2}W = 1{,}000$ lb.
Area of the cross section $= 4 \times 6 = 24$ sq. in.

$$\therefore s_h = \frac{3}{2} \times \frac{1{,}000}{24} = 62.5 \text{ lb.}$$

In the case of a circular cross section, the area above the neutral is $\pi r^2/2$, $I = \pi r^4/4$, $\bar{y} = 4r/3\pi$, and $b = 2r$. From Eq. (49),

$$s_h = \frac{(4r/3\pi) \cdot (\pi r^2/2) \cdot V}{(\pi r^4/4) \cdot 2r} = \frac{4}{3} \cdot \frac{V}{\pi r^2}$$

But πr^2 is the area of the entire cross section of the beam, and $V/\pi r^2$ is the average vertical shearing stress.

Then maximum $s_h = s_v$ is $\frac{4}{3}$ the average vertical shearing stress.

Example.—Find the maximum horizontal shearing stress in a circular beam 6 in. in diameter, carrying a central load of 10,000 lb.

Solution.—$V = \frac{1}{2}W = 5{,}000$ lb., and $A = \pi r^2 = 9\pi = 28.27$.

$$\therefore s_h = s_v = \frac{4}{3} \times \frac{5{,}000}{28.27} = 236 \text{ lb. approximately}$$

A natural question to ask is: When is there danger of a beam failing from horizontal shear before it gives way from bending? Theory shows and experiment has verified to a certain extent that when the length of a beam is less than ten times its depth, failure will occur by shear, whereas, if it is more than ten times the depth, failure will occur from bending.

Example 1.—Take a southern pine beam 6 by 10 in. and length 6 ft., carrying a uniformly distributed load of 10,000 lb. For a beam to be safe against failure from bending, it has been shown that

$$\text{B.M.} = p \times \text{section modulus}$$

But

$$\text{B.M.} = \tfrac{1}{8}Wl = \tfrac{1}{8} \times 10{,}000 \times 6 \times 12$$

$$\text{Section modulus} = \frac{1}{6}bh^2 = \frac{6 \times 10 \times 10}{6} = 100$$

$$\therefore \frac{10{,}000 \times 6 \times 12}{8} = 100p$$

$$p = 900 \text{ lb.}$$

For southern pine a safe fiber stress is 1,000 lb. Therefore, the beam is safe so far as bending is concerned. As previously explained, since the beam is a rectangle,

$$s_h = \frac{3}{2} \times \frac{V}{A}$$

But $V = \tfrac{1}{2}W = 5{,}000$ lb., and $A = 6 \times 10 = 60$.

$$\therefore s_h = \frac{3}{2} \times \frac{5{,}000}{6 \times 10} = 125 \text{ lb.}$$

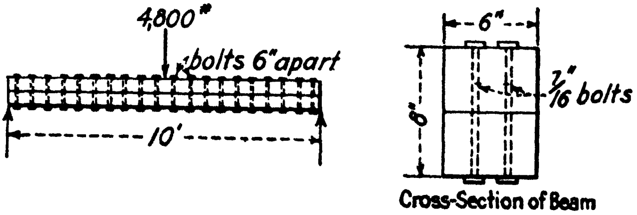
But 100 lb. per sq. in. is considered a safe horizontal shearing stress, and the beam would not be safe for the 10,000-lb. load. This example verifies the rule that when the length is less than ten times the depth, the beam will be in more danger from failure by horizontal shear than from bending, for the given stresses.

Example 2.—A beam is built up by placing two 4- by 6-in. oak timbers as shown in Fig. 352. The timbers are held together by two rows of $\frac{7}{16}$-in. bolts spaced 6 in. apart along the beam. What is the stress in the bolts?

Solution.—The bolts will be subjected to the horizontal shear resulting from the bending. The shearing resistance of two $\frac{7}{16}$-in. bolts must be equal to the total horizontal shear produced on a surface 6 in. wide and 6 in. long. The neutral surface of the built-up beam will be the surface of contact between the two beams. Then, at the neutral surface,

$$s_h = \frac{3}{2}\frac{V}{bd} = \frac{3}{2} \times \frac{4{,}800}{2 \times 6 \times 8} = 75 \text{ lb./sq. in.}$$

where $V = R_1 = 4{,}800 \div 2 = 2{,}400$.



Cross-Section of Beam

FIG. 352.

The total shear on a surface 6 by 6 in. is $75 \times 6 \times 6 = 2{,}700$ lb.
A of two $\frac{7}{16}$-in. bolts $= 2 \times 0.15 = 0.3$ sq. in.
Then

$$s_h \times 0.3 = 2{,}700 \text{ lb.}, \qquad s_h = 2{,}700 \div 0.3 = 9{,}000 \text{ lb./sq. in.}$$

Since the load is concentrated, V will have the same value for any section between R_1 and the load. The shearing stress will then be the same on all bolts.

Example 3.—In the previous example, what should be the spacing of bolts if two $\frac{3}{8}$-in. bolts are used at a section instead of the two $\frac{7}{16}$-in. bolts?

Solution.—As before, the shearing resistance of the cross sections of two bolts must be equal to the total horizontal shear on a surface 6 in. wide and having a length p, the distance between bolts, which is to be determined.

At the neutral surface,

$$s_h = \frac{3}{2}\frac{V}{bd} = \frac{3}{2} \times \frac{4{,}800}{2 \times 6 \times 8} = 75 \text{ lb./sq. in.}$$

Allowable unit shearing stress in bolts = 10,000 lb. per sq. in. A of two $\frac{3}{8}$-in. bolts $= 2 \times 0.11 = 0.22$ sq. in. Total shear to be resisted by two bolts $= 75 \times 6 \times p$; total resistance of two bolts $= 0.22 \times 10{,}000 = 2{,}200$ lb. Equating the values above, we have

$$75 \times 6 \times p = 2{,}200, \qquad p = \frac{2{,}200}{75 \times 6} = 4.88 \text{ in.}$$

As in the previous example, V is the same at all sections, and therefore the spacing just determined will be constant for the length of the beam.

Example 4.—An I beam is built up by bolting together three 2- by 8-in. planks 12 ft. long as shown in Fig. 353. What should be the spacing of ⅜-in. bolts to resist the shearing stresses?

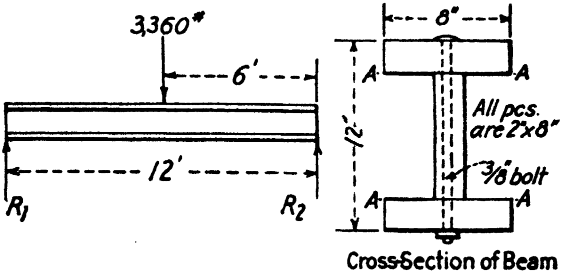


FIG. 353.

Solution.—The bolts must resist the shear at section A-A, and the general expression for shear must be used.

$$s_h = \frac{V}{Ib}\bar{y}A = \frac{3{,}360 \times 5 \times 16}{2 \times 896 \times 2} = 75 \text{ lb./sq. in.}$$

Since $V = 3{,}360/2 = 1{,}680$ lb.,

$$\begin{aligned} \bar{y} &= 4 \text{ in.} + 1 \text{ in.} = 5 \text{ in.} \\ A &= 8 \times 2 = 16 \text{ sq. in.} \\ b &= 2 \text{ in.} \\ I &= \frac{8 \times 12^3}{12} - \frac{6 \times 8^3}{12} = 896 \end{aligned}$$

The total horizontal shear on an area 2 in. wide and p in. long = $75 \times 2 \times p$.

The resistance of a ⅜-in. bolt = $0.11 \times 10{,}000 = 1{,}100$ lb.

Since the above values must be equal,

$$75 \times 2 \times p = 1{,}100$$

and the distance between the bolts

$$p = \frac{1{,}100}{150} = 7\frac{1}{3} \text{ in.}$$

Example 5.—Solve the previous example on the assumption that the load is uniformly distributed instead of concentrated.

Solution.—With the load concentrated at the middle, the shear is constant, and a single spacing will satisfy the requirements. But with the distributed load the shear changes from section to section and consequently the spacing of the bolts should change in the same way. As the shear decreases toward the middle, the spacing will increase.

The reactions will be the same as in the previous problem, and

$$V = R_1 = \frac{3{,}360}{2} = 1{,}680 \text{ lb.}$$

Then

$$s_h = \frac{V\bar{y}A}{Ib} = \frac{3{,}360 \times 5 \times 16}{2 \times 896 \times 2} = 75 \text{ lb./sq. in.}$$

which is the same as in the previous example. The spacing of the bolts would be 7⅓ in. near the supports, and this same spacing is continued for several feet. At 3 ft. from R_1, or at the quarter point,

$$V = R_1 - \frac{3{,}360 \times 3}{12} = 840 \text{ lb.}$$

since $R_1 = 3{,}360/2 = 1{,}680$ and the load per foot $= 3{,}360/12 = 280$ lb.

$$s_h = \frac{V\bar{y}A}{Ib} = \frac{840 \times 5 \times 16}{896 \times 2} = 37.5 \text{ lb./sq. in.}$$

or one-half of the value at the support, since V is one-half of the corresponding value at the support. The total horizontal shear on a length p is

$$37.5 \times 2 \times p$$

and the resistance of a ⅜-in. bolt $= 0.11 \times 10{,}000 = 1{,}100$ lb. Then

$$37.5 \times 2 \times p = 1{,}100, \qquad p = \frac{1{,}100}{37.5 \times 2} = 14\frac{2}{3} \text{ in.}$$

This spacing will be continued to the middle. In a longer beam more spacings might be computed.

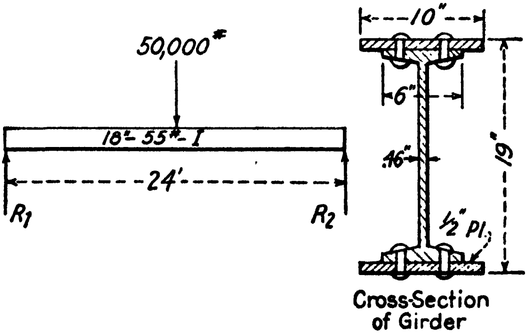


FIG. 354.

Example 6.—An 18-in. 55-lb. I beam is reinforced by riveting a 10- by ½-in. plate to each flange (Fig. 354). The span is 24 ft., and a load of 50,000 lb. is concentrated at the middle. Seven-eighths-inch rivets are to be used, for which s_s = 10,000 lb. per sq. in. and s_b = 20,000 lb. per sq. in. What should be the rivet spacing?

Solution.

$$R_1 = V = \frac{50,000}{2} = 25,000 \text{ lb.}$$

$$I \text{ of compound section} = 1,651, \qquad \bar{y} = 9.25 \text{ in.}$$

$$s_h = \frac{25,000 \times 9.25 \times 5}{1,651 \times 6} = 117 \text{ lb./sq. in.}$$

The total shear per inch of the length is 117 × 6 = 702 lb., and the total shear per foot of length is 702 × 12 = 8,424 lb.

Strength of $\frac{7}{8}$-in. rivet in bearing = $\frac{7}{8} \times \frac{1}{2} \times 20,000$ = 8,750 lb.

Strength of $\frac{7}{8}$-in. rivet in shear = 0.601 × 10,000 = 6,010 lb.

But there are two rivets at each section, and the number needed on each side is $n = 8,424/(2 \times 6,010) = 0.7$. The spacing will be p = 12 in./0.7 = 17.1 in. Since the vertical shear is constant, the spacing of 17 in. may be continued for the entire length of the beam.

148. Tension or Compression Due to Shear.—For convenience, assume the end of the prism in Fig. 348 as 1 in. square, as in Fig. 355. Pass a plane 3-3, and consider the portion to the right as a free body (Fig. 355*a*). The stresses s_s being resolved into normal and parallel components, it is seen that the normal components will produce tension on the assumed plane. Again if the plane 4-4 is passed, the normal components will produce compression on the assumed plane Fig. 355*b*. Thus it follows that *at any point in a member subjected to shear there will be found planes at right angles to each other on which either tension or compression is produced as a result of shear.*

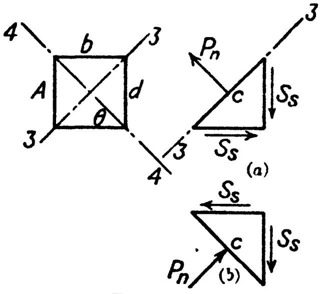


FIG. 355.

149. Maximum and Minimum Stresses.—Let the diagonal 4-4 be passed through the prism A and the left part be considered as the free body. Resolving the shearing forces perpendicular to the plane, we have

$$P_n = s_s bl \sin \Theta + s_s \, dl \cos \Theta$$

The unit compressive stress is obtained by dividing the normal force by the area of the plane.

$$s_n = \frac{P_n}{cl} = s_s \frac{cb \sin \Theta}{cl} + s_s \frac{db \cos \Theta}{cl}$$

But $b/c = \cos \Theta$, and $d/c = \sin \Theta$. Thus, the equation may be written

$$\begin{aligned} S_n &= s_s \sin \Theta \cos \Theta + s_s \sin \Theta \cos \Theta \\ &= s_s \frac{\sin 2\Theta}{2} + s_s \frac{\sin 2\Theta}{2} = s_s \sin 2\Theta \qquad (51) \end{aligned}$$

But $\sin 2\Theta$ has its greatest value when $\Theta = 45°$. Therefore, the maximum unit compressive or tensile stresses occur on planes making angles of 45° with the directions of the original shearing stresses.

150. Shearing Stress in I Beams.—The assumption is usually made in the case of I beams that the flange carries all the bending stress and the web, extending through the entire depth, takes the vertical shear. Although the assumption is not quite true, it is on the side of safety, for accordingly the designer uses more material in the flanges than is really required. Also, since the shearing value of steel is high, the web is amply able to care for the shear.

Then

$$s_v = \frac{V}{td} \qquad (52)$$

where t is the thickness of the web and d the depth of the beam.

In the design of a beam, tests should always be made for shearing strength as well as bending strength.

Example 1.—A 15-in. 75-lb. I beam carries its maximum uniformly distributed load when used as a simple beam of 20-ft. span. Find the load and the greatest unit shearing stress.

Solution.—From Table XI the section modulus is 91.6. Then

$$\tfrac{1}{8}W \times 20 \times 12 = 16{,}000 \times 91.6$$

$$W = \frac{16{,}000 \times 91.6}{30} = 48{,}850$$

Now $V = \frac{1}{2}W = 24{,}425$.

$$\text{Average } s_h = s_v = \frac{24{,}425}{15 \times 0.88} = 1{,}850 \text{ lb./sq. in.}$$

NOTE.—See Table XI for web dimensions of the beam.

A safe shearing stress is 8,000 to 10,000, which shows the stress to be well within safe limits.

Example 2.—Answer the questions of the preceding example for I beams having the following dimensions: 24-in. 115-lb.; 20-in. 90-lb.; 15-in. 42-lb.; 10-in. 25-lb.; 4-in. 7.5-lb.

151. Shear in I Beams.—A typical computation will be made to show that the method of Art 150 is substantially correct. A 10-in. 25-lb. I beam (Fig. 356) will be taken as an illustration, and the unit shear will be determined at the following points: neutral axis, base of flange, and at 4 in. from the neutral axis. These values will be expressed in terms of the total vertical shear V. Then

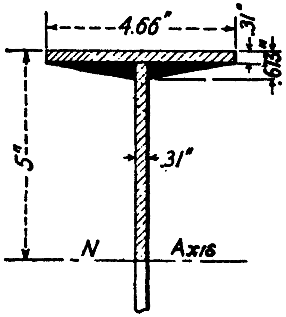


FIG. 356.

$$s_h = \frac{V}{Ib}\bar{y}A, \qquad I = 122.1, \qquad b = 0.31 \text{ in.}$$

The values of $\bar{y}A$ for the several component areas will be tabulated.

	A	$\bar{y}$	$\bar{y}A$
Horizontal rectangle	1.442	4.845	6.986
Vertical rectangle	1.455	2.345	3.412
Two triangles	0.788	4.569	3.600
Total	.....	.....	13.998

Then, at N axis,

$$s_h = \frac{V \times 13.998}{122.1 \times 0.31} = 0.370V$$

At base of flange,

$$s_h = \frac{V \times 10.586}{122.1 \times 0.31} = 0.279V$$

At 4 in. from *N* axis,

$$s_h = \frac{V \times 11.518}{122.1 \times 0.31} = 0.304V$$

The average of the three values is $0.317V$.

By the approximate method,

$$s_h = \frac{V}{10 \times 0.31} = 0.322V$$

From a comparison of these last two values, it can be seen that the equation $s_h = V/td$ gives satisfactory values.

152. Plate Girders.—In the design of the plate girder, the computation of the rivet spacing is of much importance. The preceding theory is sometimes used without any modification, and the results will be arrived at as in Example 6 of Art. 147. However, the exact theory is generally modified. The following assumptions are made: The web plate resists all the vertical shear, the flanges resist the bending, and the shearing stress is distributed uniformly over the cross section of the web. Some other approximations will be indicated.

The expression $s_h = V\bar{y}A/Ib$ may be simplified in this manner. Since the shear between the web and the flanges is to be determined, then A, which is the area beyond the shearing plane, is taken as the area of a flange. $\bar{y}$ is the distance from the center of gravity of a flange to the neutral axis. This is for practical purposes equal to one-half of the depth of the web, or $D/2$. b = the thickness of the web. $I = A\bar{y}^2 \times 2$ for two flanges. The exact value for $I = Ig + A\bar{y}^2$ for each flange. But the moment of inertia of a flange about the gravity axis of the flange is small and may be neglected in comparison with the tranfer term of the expression. If these substitutions are made, the exact formula becomes

$$s_h = \frac{V \times (D/2) \times A}{2 \times A \times (D/2)^2 \times b} = \frac{V}{D \times b}$$

The horizontal shear per inch of length is

$$\frac{V}{Db}(b \times 1) = \frac{V \times b}{D \times b} = \frac{V}{D}$$

And the horizontal shear in a length of girder equal to the depth of the web is

$$\frac{V}{D} \times D = V$$

This total amount of horizontal shear is seen to be equal to the total vertical shear. A length equal to D is therefore taken as the unit and a uniform rivet spacing determined for this unit. Then in any unit length there must be a sufficient number of rivets connecting the flange angles to the web plate to offer a resistance equal to the maximum V in that unit.

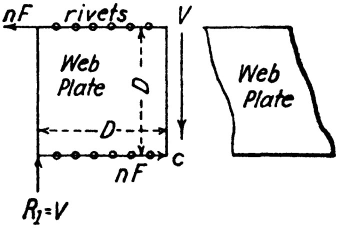


FIG. 357.

Since the rivets are in bearing and in shear, the least of these values will be the strength of a rivet. Let this strength of a rivet be designated by F. Then the number of rivets joining each flange with the web in a length equal to the depth of the web will be $n = V/F$, and the spacing is given by

$$p = \frac{D}{n}, \qquad \text{or } p = \frac{D}{V/F} = \frac{FD}{V} \tag{53}$$

The diameter of the rivets is always assumed.

The simplified forces acting in accordance with the equations just derived are shown in Fig. 357 where a unit length is taken as a free body. It is seen that the vertical shears constitute the overturning couple, and the resistance of the rivets provides the resisting couple. Then the relationship of the forces resolves itself into a moment equation. Taking moments about c, we have

$$nF \times D = V \times D, \qquad \text{or } nF = V$$

If the vertical shear is constant at all sections, then a single computation of rivet spacing will hold for the full length of the girder; but if the shear varies from section to section, then the maximum shear acting on any unit is assumed to be constant for that unit.

Example 1.—A plate girder 36 ft. long is used as a simple beam (Fig. 358). The web is 36 by ⅜ in. The flange angles are 6 by 6 by ⅜ in. The girder supports a total load of 240,000 lb. at the middle of the span. What is the spacing for ⅞-in. rivets? For rivets, $s_s = 10,000$ lb./sq. in. $s_b = 20,000$ lb./sq. in.

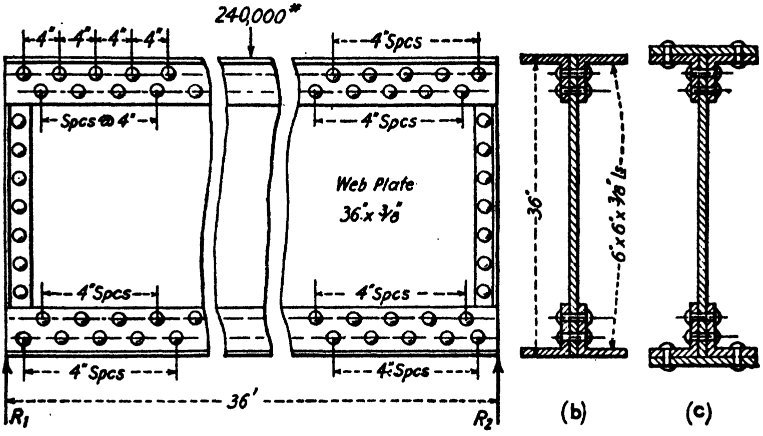


FIG. 358.

Solution.—$V = R_1 = 240,000/2 = 120,000$ lb. Area of ⅞-in. rivet $= 0.601$ sq. in.

Resistance of ⅞-in. rivet in bearing $= \frac{7}{8} \times \frac{3}{8} \times 20,000 = 6,550$ lb.

Resistance of ⅞-in. rivet in double shear =

$$2 \times 0.601 \times 10,000 = 12,020 \text{ lb.}$$

Then

$$p = \frac{6,550 \times 36}{120,000} = 2 \text{ in.}$$

This spacing would be a close one, but the rivets can be arranged in two rows, with a 4-in. spacing in each row and with rivets staggered. Since the shear is constant, the spacing will be uniform for the length of the girder.

Example 2.—In the previous example, let the load be uniformly distributed and all other data the same. What is the rivet spacing?

Solution.—$V = R_1 = 240{,}000/2 = 120{,}000$ lb. For the first unit the rivet spacing will be as in Example 1. It is assumed that V is constant for a length of 36 in. For the second unit a new V must be obtained. The uniform load per foot is $w = 240{,}000/36 = 6{,}667$ lb Then

$$V = 120{,}000 - 3 \times 6{,}667 = 100{,}000 \text{ lb. at 36 in. from the left end}$$

$$p = \frac{6{,}550 \times 36}{100{,}000} = 2.36 \text{ in.}$$

Two rows of rivets with twice the spacing, or 4.72 in. or 4¾ in., would be used for a distance of 36 in. The next $V = 120{,}000 - 6 \times 6.667 = 80{,}000$ lb. at 72 in. from the left end. Then

$$p = \frac{6{,}550 \times 36}{80{,}000} = 2.95 \text{ in.}$$

and again with two rows the rivets in each row would be spaced 2 × 2.95 in., or 6 in., which would be the maximum pitch permitted by the specifications, and no further spacing need be computed.

In Fig. 358*c*, cover plates 10⅜ by ½ in. are added to the girder, and these must be riveted to the flange angles. The total shear is transmitted from the web plate to the flange. Each component part of the flange is assumed to offer a resistance corresponding to its own proportion of the total flange area. Thus, the resistance of the plate is transmitted through the rivets to the flange angles and then to the web.

In the example just solved, $V = 120{,}000$ lb. The area of a 6- by 6- by ⅜-in. angle is 4.36 sq. in. The area of the plate is 10⅜ by ½ in. = 5.19 sq. in. The area of the flange = $(4.36 \times 2) + 5.19 = 13.91$ sq. in. The plate must transmit $(5.19/13.91) \times 120{,}000 = 44{,}750$ lb. Since there must be two rows of rivets, the pitch required is $p = (6{,}010 \times 36 \times 2)/44{,}750 = 9.65$ in. This pitch is much in excess of the pitch for the flange riveting and is also larger than the maximum allowable pitch of 6 in. The pitch of 6 in. will then be used unless there is interference with the flange rivets, in which case the cover-plate rivets are staggered with the flange rivets. However some specifications permit a rivet spacing in excess of 6 in. and in such cases the design is governed by these specifications.

Problems

283. A wooden beam 6 by 8 in. and 12 ft. long, supported at the ends, carries a load of 2,000 lb. at the middle of the span. What is the unit shearing stress at the neutral axis? At a point 3 ft. from the end and 2 in. from the neutral axis? What is the bending stress in the beam?

284. A simple beam 6 by 6 in. and 6 ft. long carries a concentrated load at the middle of the span. What is the load if it is determined by the allowable unit shearing stress of 120 lb. per sq. in.? What is the deflection if E is 1,200,000?

285. What is the maximum unit shearing stress in the beam of Example 6, Art. 142. What is the unit shearing stress 1 in. above the bottom of the beam?

286. What is the maximum unit shearing stress in Example 5, Art. 142. What will the shearing unit stress be 2 in. below the top of the beam?

287. A rectangular wooden beam 8 by 12 in. with a 12-ft. simple span carries a uniform load of 1,600 lb. per ft. What is the unit horizontal shearing stress at the following points: at the neutral axis and at the end of the beam; 4 in. from the neutral axis and 1 ft. from the support; 2 in. from the neutral axis and at a quarter point?

288. What is the maximum unit shearing stress developed in a 10-in. 30 lb. I beam that carries a uniform load over a simple span of 12 ft. if the fiber stress due to bending is 16,000 lb. per sq. in.

289. What is the maximum unit shearing stress in a timber beam 10 by 14 in. with a simple span of 9 ft. if loads of 12,000 lb. each are applied at the one-third points?

290. What will be the total uniform load that can be placed on a southern-pine beam 10 by 14 in. and 8 ft. long, supported at the ends, if the allowable shearing stress is 110 lb. per sq. in.? What will be the stress due to bending?

291. If the maximum allowable unit shearing stress is 120 lb. per sq. in., determine the length of a 6- by 12-in. beam for a simple span above which bending will govern and below which shear governs. Assume that the beam carries a uniform load.

292. If the unit shearing stress is assumed at 120 lb. per sq. in. and the bending stress shall not exceed 1,200 lb. per sq. in., what should be the proper length of a 10- by 12-in. wooden beam carrying a uniform load of 1,800 lb. per lin. ft.? Solve for both a simple span and a cantilever.

293. What should be the spacing of ½-in. bolts in a beam consisting of a 4- by 6-in. oak timber bolted to a 6- by 6-in. timber? The pieces will be placed so that the cross section of the beam will be 6 by 10 in. The span and loading are shown in Fig. 352.

294. A box girder is built up by bolting together four 2- by 10-in. planks 10 ft. long, as in Fig. 359. If the bolts are spaced 9 in. apart, what size of bolt would be suitable and how many bolts are needed?

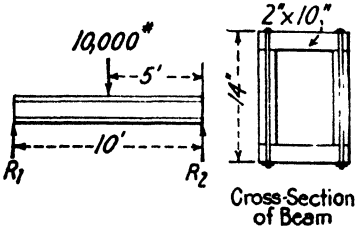


FIG. 359.

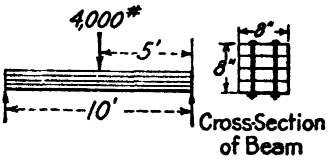


FIG. 360.

295. A simple beam 10 ft. long is built up by placing four 2- by 8-in. planks on top of each other to form a beam 8 by 8 in. in cross section. How many ½-in. bolts arranged in two rows are needed? The allowable shearing stress on a bolt is 10,000 lb. per sq. in. The loading is shown in Fig. 360.

296. A plate girder used as a simple beam 36 ft. long must support a concentrated load of 180,000 lb. at the middle of the span and 45,000 lb. at each quarter point (Fig. 361). The girder is composed of a 48- by ⅜-in. web plate and four 6- by 6- by ¾-in. flange angles. What should be the rivet spacing if ⅞-in. rivets are used? The allowable unit shearing stress on the rivets is 12,000 lb. per sq. in., and the allowable bearing stress is 20,000 lb. per sq. in.

297. What would be the rivet spacing in Prob. 296, if 1-in. rivets were used instead of ⅞-in. rivets?

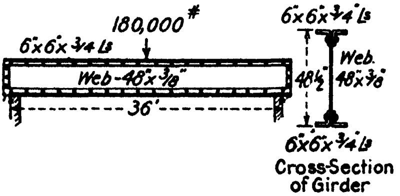


FIG. 361.

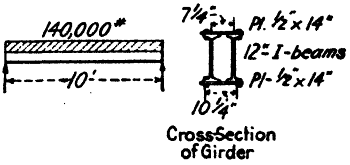


FIG. 362.

298. A beam box girder has a span of 10 ft. It supports a uniformly distributed load of 140,000 lb. What should be the spacing of ⅞-in. rivets. The details of the beam are shown in Fig. 362.

299. In Example 3, Art. 147, if the bolts are spaced 6 in. apart, what size of bolts will be needed? The allowable shearing stress in a bolt is 10,000 lb. per sq. in.

300. In the I beam of Example 4, Art. 147, what should be the size of bolts if the spacing is 9 in. and the allowable shearing stress in a bolt is 10,000 lb. per sq. in.?

301. In Example 4, Art. 137, what will be the maximum unit horizontal shear?

302. What will be the maximum shearing stress in the beam of Example 5, Art. 137.

303. A plate girder 24 ft. long with a web plate 30 by ½-in. carries a uniformly distributed load of 180,000 lb. What pitch is required at each end and at the quarter point if 1-in. rivets are used? The allowable shearing stress is 10,000 lb. per sq. in., and the bearing stress is 20,000 lb. per sq. in.

304. In Prob. 296, let a ⅜- by 12-in. flange plate be riveted to the flange angles at the top and the bottom. How far apart should ⅞-in. rivets be placed?

305. In Prob. 283, what will be the unit tensile stress at the neutral axis, due to the shearing stresses? What will be the unit tensile stress caused by the shear at a point 2 in. from the neutral axis?

306. In Prob. 287, what unit tensile stress will result from shear at the quarter point of the beam and 2 in. from the neutral axis?

307. A 12-in. 65-lb. wide flange beam is used for a simple span of 12 ft., and carries a uniformly distributed load. The allowable bending stress is 12,000 lb. per sq. in. What is the maximum unit shearing stress at the base of the upper flange? Consider the flange as a rectangle.

CHAPTER XIV

DEFLECTION, TYPICAL CASES

153. Discussion.—When a load is placed on a beam, experiment shows that every portion of the beam is moved in a direction parallel to the load. The distance that a particle is moved increases with the distance of the particle from the support until a maximum value is reached. The distance that a portion moves is called *deflection.* The neutral surface becomes a curved surface, and its intersection with a vertical plane drawn through the center of the beam lengthwise is called the *elastic curve.* Using the initial position of the neutral axis as the x axis, we find that the ordinate of any point on the elastic curve is the deflection of the beam at that point. Deflection plays a very important part in the design of structures and machines. If floor beams or joists deflect too far, the plaster on the ceiling under them may crack and ruin its appearance, although no damage to the structure may result. Also, a floor supported by such beams may be so out of level that its usefulness may be impaired, especially for machines such as a planer or lathe.

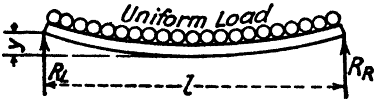


FIG. 363.

Before the reactions, bending moment, and shear for continuous beams and beams with fixed ends can be found, the theory of deflection must be developed.

154. Radius of Curvature in Terms of Bending Moment. By the *radius of curvature* at any point on a curve is understood the radius of the circle drawn tangent to the curve at the point. That is, the radius is perpendicular to the tangent to the curve at the point. In general, the elastic curve is not a circle. But, for a very short length such as $DE = \Delta l$, it may be considered as the arc of a circle (Fig. 364). Before the load was placed on the beam, *HB* was in the position *GC* parallel to *AF*. After the loading, they make an angle θ with each other. However, they remain perpendicular to the neutral axis or the elastic curve. They are radii of the same circle, and their intersection *O* is the center of curvature. Then $DO = EO = r$ is the radius of curvature. The bending of the beam lengthens fiber *FG* to *FH*. Then

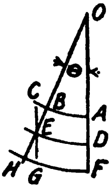


Fig. 364.

$$GH = \lambda$$

$$\delta = \frac{GH}{\Delta l} = \frac{\lambda}{\Delta l} \qquad (g)$$

Let *s* be the unit stress in the outside fiber *FG*. Then bending moment

$$M = \frac{sI}{\overline{EG}} \qquad \text{or } s = \frac{M \times \overline{EG}}{I} \qquad (h)$$

Also,

$$E = \frac{s}{\delta} = \frac{(M \times \overline{EG})/I}{\lambda/\Delta l} = \frac{M \times \overline{EG} \times \Delta l}{\lambda I} \qquad (k)$$

By similar triangles *ODE* and *EGH*,

$$OD:DE::EG:GH$$

or

$$r:\Delta l::EG:\lambda$$

$$r = \frac{\Delta l \times \overline{EG}}{\lambda}$$

Substituting for *r* in (*k*) above and solving,

$$r = \frac{EI}{M}$$

In Fig. 364, it is evident that the more DE is bent the greater the deflection becomes, while the radius of curvature gets smaller. That is, the deflection varies as the reciprocal of r.

$$\text{Deflection} = \frac{kM}{EI}$$

The deflection is found to depend directly on M and inversely on EI.

In Chap. XII, a rule was given for determining the sign of the bending moment. This rule is somewhat confusing and a better one may be had by noting the center of curvature of the bent beams. Thus, if the center of curvature is below the beam, the moment is taken as negative, as in the case of the cantilever beam. If the center of curvature is above the beam, the moment is taken as positive, as in the case of the simple beam.

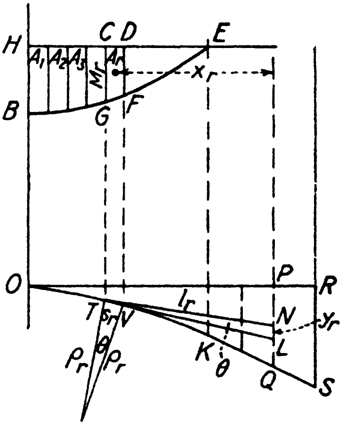


FIG. 365.

155. Formulas for Deflection.—Let OR (Fig. 365) be the neutral axis of a cantilever beam, with R the free end carrying no load. After a load is applied, the neutral axis makes the curve $OKQS$ with the bending greatly exaggerated. In practice, the assumption is made that the point R moves downward along a vertical line, so that the deflection of the point R is RS. It is now proposed to find the deflection PQ of any point P.

Let $BCDE$ be the moment diagram. Divide OK into small parts, one of which is TV. Notice that KS is a straight line. At T and V, draw the radii of curvature ρ_r forming the angle Θ; also, draw the tangents TN and VL, which also form the angle Θ with each other. Since the arc $TV = s_r$ is very small, $TN = VL = l_r = x_r$. Now the bending of the beam T to V caused point P to deflect through the distance $NL = y_r$. Also,

$$y_r = l_r\Theta$$

$$\Theta = \frac{s_r}{\rho_r}, \qquad \text{and } \rho_r = \frac{EI}{M_r}$$

Then

$$y_r = \frac{l_r s_r}{\rho_r} = \frac{l_r s_r M_r}{EI}$$

But M_r is the bending moment at T and is equal to GC of the moment diagram. Also, s_r is practically equal to GF. s_rM_r may then be taken for the area $GCDF = A_r$.

$$\therefore y_r = \frac{l_r A_r}{EI} = \frac{x_r A_r}{EI}$$

The product x_rA_r is the moment of the area A_r with respect to the point P of the beam.

It is now evident that the deflection PQ is the sum of the deflections y_r caused by the bending of the small parts into which the beam was divided. By analogy,

$$y_1 = \frac{x_1A_1}{EI}, \qquad y_2 = \frac{x_2A_2}{EI}, \qquad \cdots \qquad y_n = \frac{x_nA_n}{EI}$$

$$\begin{aligned} PQ &= y = y_1 + y_2 + y_3 + \cdots + y_n \\ &= \frac{1}{EI}(x_1A_1 + x_2A_2 + x_3A_3 + \cdots + x_nA_n) \\ &= \frac{1}{EI}\Sigma x_rA_r \end{aligned}$$

From Chap. X on Center of Gravity and Stability,

$$\begin{aligned} x_1A_1 + x_2A_2 + \cdots + x_nA_n &= \bar{x}(A_1 + A_2 + \cdots + A_n) \\ &= \bar{x}A \end{aligned}$$

$$\therefore y = \frac{\bar{x}A}{EI} \tag{54}$$

where A is the entire area under the moment diagram and $\bar{x}$ is the distance to its center of gravity from the point where the deflection is to be found. The theorem is as follows: *The deflection of a point P on a beam above or below the point O, where the tangent is horizontal, is equal to the static moment of the area under the moment diagram between O and P, about the point P, divided by EI.*

156. Cantilever with a Concentrated Load on the Free End.—Figure 366 shows the various diagrams. a is the beam of effective span l. Although the beam is shortened by the bending, it is so small that the full length is taken. b is the shear diagram, c the moment diagram, and d the elastic curve. The maximum deflection is at the free end of the beam and is represented by $y_{\max}$. C is the center of gravity of the moment area. By the theorem,

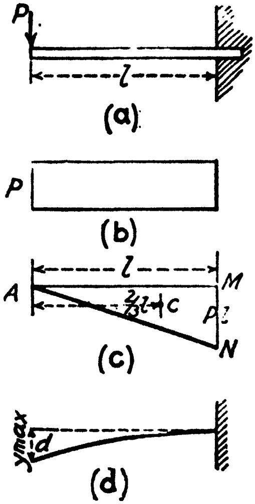


FIG. 366.

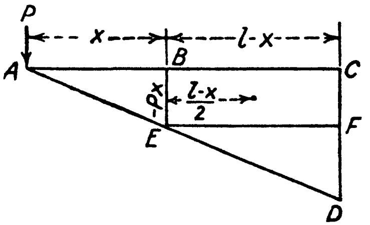


FIG. 367.

$$y_{\max} = -\frac{1}{EI} \times \frac{2}{3}l \times \frac{l}{2} \times Pl$$

$$y_{\max} = -\frac{Pl^3}{3EI} \qquad (55)$$

The deflection at any point of this beam is found as follows: In Fig. 367, let B be the point at distance x from the free end. Divide the area to the right of B into a rectangle and triangle.

$$\text{Area } BCFE = -Px(l - x)$$

$$\text{Moment arm} = \frac{l - x}{2}$$

$$\text{Moment } BCFE = -\frac{Px}{2}(l-x)^2$$

$$CD = -Pl, \qquad CF = -Px, \qquad FD = -P(l-x)$$

$$\text{Area } EFD = -\tfrac{1}{2}P(l-x)^2$$

$$\text{Moment arm} = \tfrac{2}{3}(l-x)$$

$$\text{Moment } EFD = -\frac{P}{3}(l-x)^3$$

Total moment equals

$$-\frac{Px}{2}(l-x)^2 - \frac{P}{3}(l-x)^3 = -\frac{Pl^3}{3} + \frac{Pl^2x}{2} - \frac{Px^3}{6} \qquad (m)$$

$$\therefore \text{ Deflection at } B = y = -\frac{P}{6EI}(2l^3 - 3l^2x + x^3) \qquad (55a)$$

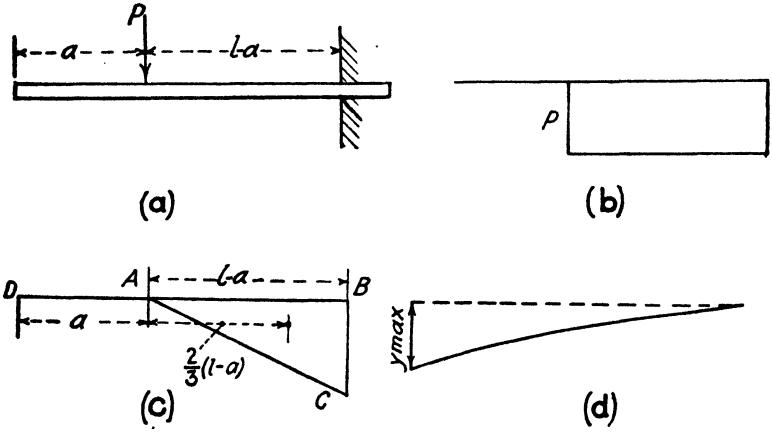


FIG. 368.—(*a*) and (*c*) moment diagrams; (*b*) shear diagram; (*d*) elastic curve.

When $x = 0$, B is the free end, and

$$y = -\frac{P}{6EI}(2l^3) = -\frac{Pl^3}{3EI}$$

which agrees with $y_{\max}$ previously found. Also, when $x = l$, $y = 0$.

157. Cantilever with a Load Not at the End.—Figure 368 shows a load P at the distance a from the free end, and the various diagrams.

$$\text{Area } ABC = -\frac{P}{2}(l-a)^2$$

$$\text{Moment arm} = a + \tfrac{2}{3}(l-a)$$

$$= \frac{2l+a}{3}$$

$$\text{Moment about } D = -\frac{2l+a}{2} \times \frac{P}{2}(l-a)^2$$

$$\therefore\ y_{\max} = -\frac{P}{6EI}(2l^3 - 3l^2a + a^3) \qquad (55a)$$

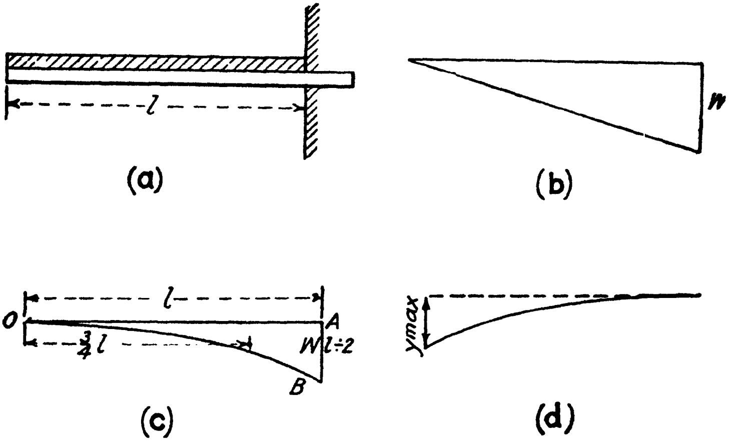


FIG. 369.—(*a*) and (*c*) moment diagrams; (*b*) shear diagram; (*d*) elastic curve.

If $x = a$ is put in Eq. (*m*) above, this becomes Eq. (55*a*). These equations show that the deflection at any point B of a cantilever with a load on the free end is equal to the deflection of the free end due to the same load at B.

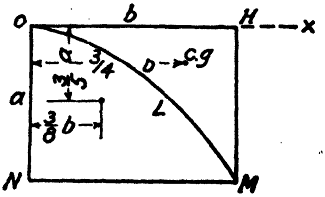


FIG. 370.

158. Cantilever with a Uniformly Distributed Load.—The several diagrams are shown in Fig. 369. The curve of the moment diagram *c* is a parabola. The area OAB for which the static moment is to be found is between the x axis and the curve.

It is proved in mathematics that for the half segment of a parabola shown in Fig. 370 the area of the parabolic segment *OMN* is two-thirds the area of the rectangle *OHMN*. Then area *OHML* equals $\frac{1}{3}OHMN$, or

$$OHML = \tfrac{1}{3}ab$$

Also, the coordinates of the center of gravity of *OLMN* are $\frac{3}{5}ON$ and $\frac{3}{8}OH$ or $\frac{3}{8}NM$.

The center of gravity of *OHML* must next be found.

$$\bar{x} = \frac{ab \times (b/2) - \frac{2}{3}ab \times \frac{3}{8}b}{ab - \frac{2}{3}ab} = \frac{3}{4}b.$$

That is, for the moment area, $\bar{x} = \frac{3}{4}l$. Also,

$$\text{Area } OAB = -\frac{1}{3} \times l \times \frac{Wl}{2} = -\frac{Wl^2}{6}$$

Then

$$y_{\max} = -\frac{(Wl^2/6) \times \frac{3}{4}l}{EI} = -\frac{Wl^3}{8EI} \qquad (56)$$

159. Simple Beam with a Concentrated Load at the Center.—The different diagrams for this beam are shown in Fig. 371. By the theorem, with *A* as the point deflected upward above the horizontal tangent,

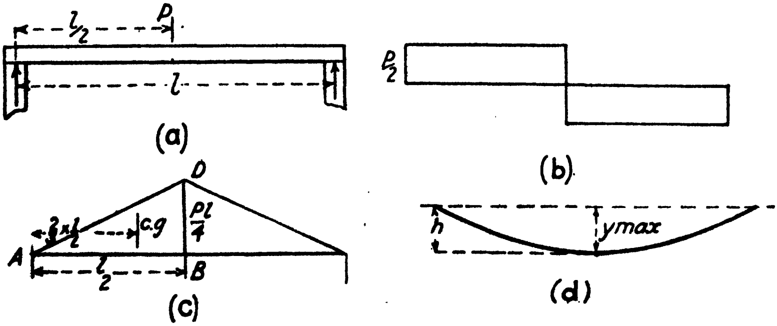


FIG. 371.—(*a*) and (*c*) moment diagrams; (*b*) shear diagram; (*d*) elastic curve.

$$h = +\frac{l}{3} \times \frac{ABD}{EI}$$

$$= +\frac{l}{3EI} \times \frac{1}{2} \times \frac{l}{2} \times \frac{Pl}{4}$$

$$= +\frac{Pl^3}{48EI}$$

But the deflection $y_{\max}$ of the beam is opposite in sign to h.

$$\therefore\ y_{\max} = -\frac{Pl^3}{48EI} \qquad (57)$$

160. Simple Beam with a Uniformly Distributed Load.—The diagrams are shown in Fig. 372. The moment diagram (*c*) is a parabola with its vertex at O. The area AOB is $\frac{2}{3}$ of $ABOD$, or

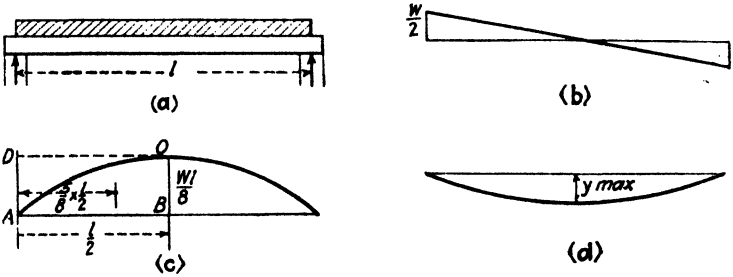


FIG. 372.—(*a*) and (*c*) moment diagrams; (*b*) shear diagram; (*d*) elastic curve.

$$\text{Segment } AOB = \frac{2}{3} \times \frac{l}{2} \times \frac{Wl}{8} = \frac{Wl^2}{24}$$

In Art 158, it was given that $\bar{x} = \frac{3}{8}BA = \frac{3}{16}l$ for the segment AOB. Then, with A as the moment center, the x coordinate of the center of gravity is

$$\bar{x}_0 = \frac{5}{8} \times \frac{l}{2} = \frac{5}{16}l$$

Then

$$y_{\max} = -\frac{1}{EI}\frac{5}{16}l \times \frac{Wl^2}{24} = -\frac{5}{384}\frac{Wl^3}{EI} \qquad (58)$$

In all these forms for deflection, l is expressed in inches. Then the deflection is in inches.

The deflection of a beam is a function of the stiffness of the beam.

In buildings, the deflection is usually limited to $\frac{1}{360}$ of the length of the beam in inches.

161. Simple Beam with a Load Not at the Center.—In this case the formula for deflection is more difficult to derive, for the maximum deflection does not occur under the concentrated load. However, if the concentrated load is considered as acting at the center of the beam and the deflection found for this case, the result will be greater than the actual deflection. But a beam designed by this hypothesis will be on the side of safety.

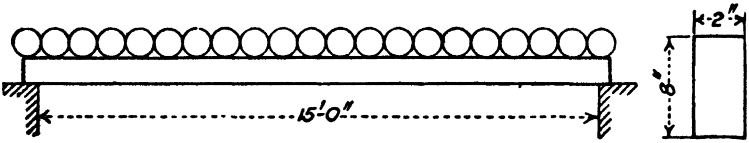


FIG. 373.

Example 1.—Suppose that a wooden beam 2 by 8 in. is 15 ft. long and carries a uniformly distributed load of 800 lb. (Fig. 373). To find if the timber is strong enough and at the same time stiff enough.

Solution.

$$\frac{1}{8}Wl = \frac{p \times b \times h^2}{6}$$

Here $W = 800$, $l = 15 \times 12 = 180$, $b = 2$, $h = 8$, and p is the fiber stress developed. Substituting these values in the formula,

$$\frac{1}{8} \times 800 \times 180 = \frac{p \times 2 \times 8 \times 8}{6}$$

$$p = \frac{6 \times 18{,}000}{128} = 844 \text{ lb./sq. in.}$$

If a safe fiber stress for timber is 1,000 lb., since the beam is not stressed so high a 2- by 8-in. beam is amply strong.

For the deflection $W = 800$, $l = 180$, $E = 1{,}200{,}000$, and

$$I = \frac{bh^3}{12} = \frac{2 \times 8 \times 8 \times 8}{12} = 85.3$$

Substituting in Eq. (58)

$$y = \frac{5 \times 800 \times 180 \times 180 \times 180}{384 \times 1{,}200{,}000 \times 85.3}$$

$$y = 0.6 \text{ in. (approximately)}$$

By the rule that the deflection must not exceed $\frac{1}{360}$ of the length in inches, it is seen that the allowable sag is

$$\tfrac{1}{360} \times 180 = \tfrac{1}{2} = 0.5 \text{ in.}$$

Consequently, the beam though strong enough is not stiff enough. If the load that this beam supports did not all come on the floor at once, the 2- by 8-in. beam might do, but safety would require a larger beam.

Try a 2- by 10-in. beam of the same length.

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12} \times 2 \times 10 \times 10 \times 10 = 166\tfrac{2}{3}$$

Since every letter has the same value for the 2- by 8-in. beam except I and since $166\frac{2}{3}$ is roughly twice $85\frac{1}{3}$, the deflection for the 2- by 10-in. beam will be approximately one-half that for the 2- by 8-in. beam, or $d = 0.3$ in. This value is evident, for I is below the line in the fraction; and since the divisor in the second case is twice that in the first case, the result in the second will be one-half of the first. Since 0.3 in. is less than 0.5 in., the 2- by 10-in. beam will be stiff enough. Also, if a 2- by 8-in. beam is strong enough to hold 800 lb., a 2- by 10-in. beam will also hold the load safely.

Example 2.—Select an I beam 15 ft. long, both strong and stiff enough to carry a section of floor 8 ft. wide on which there is a uniform load of 100 lb. per sq. ft. of surface (Fig. 374).

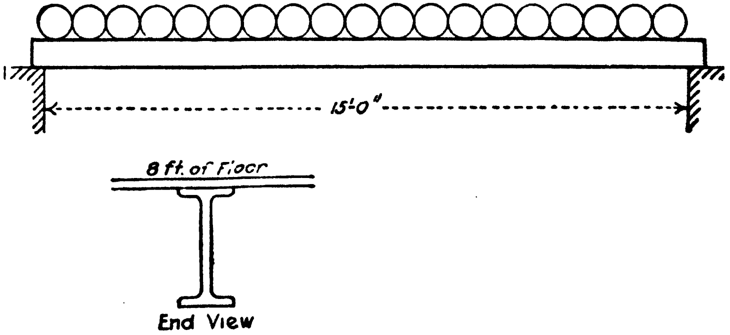


FIG. 374.

Solution.—The area of the floor is $8 \times 15 = 120$ sq. ft. Then the uniform load on the beam is

$$W = 120 \times 100 \text{ lb.} = 12{,}000 \text{ lb.}$$

Now $\frac{1}{8}Wl = p \times$ section modulus.
Here $l = 15 \times 12 = 180$ in., and $p = 16{,}000$ lb./sq. in.
Substituting these values in the equation, we have

$$\tfrac{1}{8} \times 12{,}000 \times 180 = 16{,}000 \times \text{section modulus}$$
$$\therefore \text{Section modulus} = 17 \text{ (nearly)}$$

From a handbook or Table XI, the I beam, 9-in. – 21.8 lb., will have a large enough section modulus.

For the deflection of the steel beam, $l = 180$, $W = 12{,}000$, $E = 29{,}000{,}000$, and I from the table is 84.9. Substituting in Eq. (58) produces

$$y = \frac{5 \times 12{,}000 \times 180 \times 180 \times 180}{384 \times 29{,}000{,}000 \times 84.9}$$
$$y = \frac{5 \times 12 \times 18 \times 18 \times 18}{384 \times 29 \times 84.9} = 0.37 \text{ in. (nearly)}$$

The allowable deflection is $\frac{1}{360} \times 15 \times 12 = 0.5$ in. Since the beam sags 0.37 in. and is less than 0.5 in., it is both strong and stiff enough.

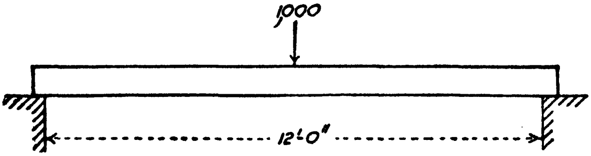


FIG. 375.

Example 3.—Take a wooden beam 4 by 10 in., of length 12 ft. To find the sag when a concentrated load of 1,000 lb. is placed at the center (Fig. 375).

Solution.

$$I = \frac{1}{12}bh^3 = \frac{4 \times 10 \times 10 \times 10}{12} = 333.3$$
$$W = 1{,}000, \qquad l = 12 \times 12 = 144, \qquad \text{and } E = 1{,}200{,}000$$

Substituting in Eq. (57)

$$y = \frac{1{,}000 \times 144 \times 144 \times 144}{48 \times 1{,}200{,}000 \times 333} = 0.16 \text{ in.}$$

Then the beam is stiff enough.

To determine the strength of the beam, we use

$$\text{B.M.} = p\frac{bh^2}{6}$$

For a simple beam with a concentrated load at the center, bending moment is

$$\frac{1}{4}Wl = \frac{1{,}000 \times 12 \times 12}{4} = 36{,}000 \text{ in.-lb.}$$

$$\therefore\ 36{,}000 = \frac{p \times 4 \times 10 \times 10}{6}$$

$$p = 540 \text{ lb./sq. in.}$$

But a safe stress is 1,000 lb.; consequently, the beam has sufficient strength.

162. Combined Loads.—When a beam has both a concentrated load and a uniformly distributed load, the total deflection is obtained by calculating the deflection due to each load separately and adding the results. Such a case arises when floor beams support a uniform floor load and a heavy piece of machinery. The sag due to the machine is greatest when the load is at the center. Then, if the beam is stiff enough with the machine on the center of the beam, it will be safe in any position.

Example—Wooden floor beams 6 by 12 in., 15 ft. long, spaced 2 ft. from center to center, support a uniform floor load of 50 lb. per sq. ft. and a machine weighing 800 lb. To find the deflection of any beam with this machine at the center of the member.

Solution.—The floor area carried by a beam is 2 × 15 = 30 sq. ft. Then the load is 30 × 50 = 1,500 lb. = W, $l = 12 \times 15 = 180$ in.

$$E = 1{,}200{,}000$$

and

$$I = \frac{1}{12}bh^3 = \frac{6 \times 12 \times 12 \times 12}{12} = 864$$

Substituting in Eq. (58) for distributed load,

$$y = \frac{5 \times 1{,}500 \times 180 \times 180 \times 180}{384 \times 1{,}200{,}000 \times 864}$$

$$y = \frac{5 \times 15 \times 18 \times 18 \times 18}{384 \times 12 \times 864} = 0.11 \text{ in.}$$

For the sag due to the load of 800 lb. = W, substitute in Eq. (57),

$$y = \frac{800 \times 180 \times 180 \times 180}{48 \times 1{,}200{,}000 \times 864}$$

or

$$y = \frac{8 \times 18 \times 18 \times 18}{48 \times 12 \times 864} = 0.1 \text{ in. nearly}$$

Then the total deflection $d = 0.11 + 0.1 = 0.21$ in.

The safe deflection is $\frac{1}{360} \times 180 = 0.5$. The beam is sufficiently stiff to carry the load.

The bending moment for the distributed load is $\frac{1}{8}Wl = \frac{1}{8} \times 1{,}500 \times 180 = 33{,}750$ in.-lb.

The bending moment for the concentrated load is $\frac{1}{4}Wl = \frac{1}{4} \times 800 \times 180 = 36{,}000$ in.-lb.

The total bending moment is $33{,}750 + 36{,}000 = 69{,}750$ in.-lb. But B.M. $= p \times$ section modulus or,

$$69{,}750 = p \times \frac{bh^2}{6}$$

$$69{,}750 = p \times \frac{6 \times 12 \times 12}{6} = 144p$$

$$p = 485 \text{ lb./sq. in.}$$

But 1,000 lb. is a safe bending stress. Therefore, the 6- by 12-in. beams are safe, both in strength and stiffness.

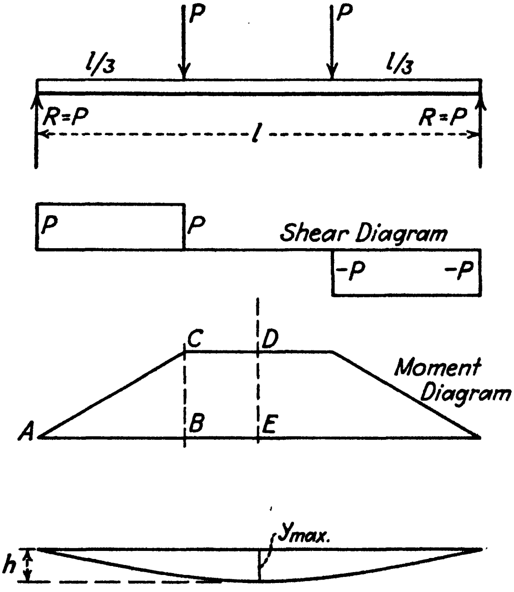


FIG. 376.

163. Simple Beam with Equal Concentrated Loads at the Third Points.—The several diagrams are shown in Fig. 376. By the theorem, the point A may be considered to be deflected upward from the horizontal tangent to the mid-point of the curve. One-half of the area of the moment diagram will accordingly be involved. Then

$$h = +\frac{2}{3} \times \frac{l}{3} \times \frac{ABC}{EI} + \left(\frac{l}{3} + \frac{1}{2} \times \frac{l}{6}\right)\frac{BCDE}{EI}$$

$$h = +\frac{2}{9}\frac{l}{EI} \times \frac{l}{3} \times \frac{Pl}{3} \times \frac{1}{2} + \frac{5}{12} \times l \times \frac{l}{6} \times \frac{Pl}{3} \times \frac{1}{EI}$$

$$h = +\frac{Pl^3}{81EI} + \frac{5Pl^3}{216EI} = \frac{621Pl^3}{17{,}496EI} = \frac{23Pl^3}{648EI}$$

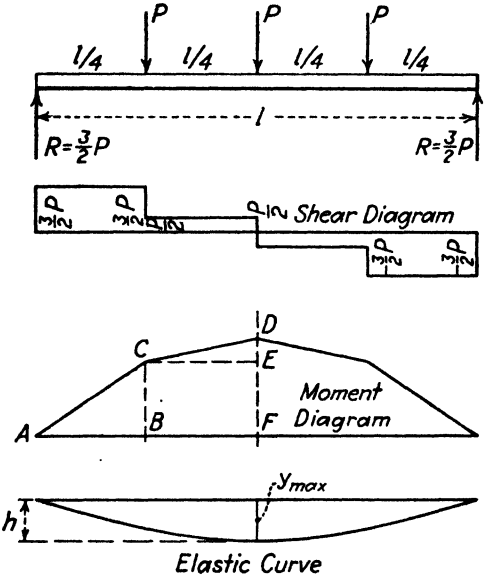


FIG. 377.

But $y_{\max}$ of the beam is opposite in sign to h.

$$\therefore\ y_{\max} = -\frac{23Pl^3}{648EI} \tag{59}$$

164. Simple Beam with Equal Concentrated Loads at the Quarter Points.—The diagrams are shown in Fig. 377. As before, the point A may be considered to be deflected upward from the horizontal tangent to the mid-point of the curve. Then

$$h = \frac{2}{3}\times\frac{l}{4}\times\frac{ABC}{EI} + \left(\frac{l}{4}+\frac{1}{2}\times\frac{l}{4}\right)\frac{BCEF}{EI} + \left(\frac{l}{4}+\frac{2}{3}\times\frac{l}{4}\right)\frac{CDE}{EI}$$

$$h = \frac{1}{EI}\left(\frac{l}{6}\times\frac{3Pl^2}{64}+\frac{3}{8}l\times\frac{3Pl^2}{32}+\frac{51}{12}\times\frac{Pl^2}{64}\right)$$

$$h = \frac{1}{EI}\left(\frac{Pl^3}{128}+\frac{9Pl^3}{256}+\frac{5Pl^3}{768}\right)=\frac{1}{EI}\left(\frac{38Pl^3}{768}\right)=\frac{19Pl^3}{384EI}$$

But the deflection $y_{\max}$ of the beam is opposite in sign to h.

$$\therefore\ y_{\max} = -\frac{19Pl^3}{384EI} \qquad (60)$$

165. Deflection Due to Oblique Loading.—If a load or a force is applied to a beam so as to produce bending simultaneously about two axes, the loading is said to be *oblique*. In order to find the total deflection the values of the deflections due to bending about each principal axis are first obtained. Since these deflections occur in directions at right angles to each other, the total value or resultant deflection will be found by means of the triangle law. The identical solution is used if the beam is subjected to vertical and horizontal forces simultaneously.

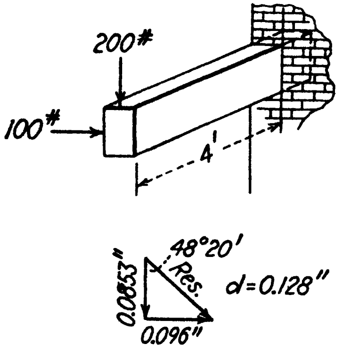


FIG. 378.

Example.—A 4- by 6-in. cantilever beam, 4 ft. long, supports a load of 200 lb. at the end (Fig. 378). There is also a horizontal force of 100 lb. applied at the end at right angles to the beam. Find the resultant deflection.

With respect to axis 1-1,

$$d = \frac{200\times 48^3}{3\times 1{,}200{,}000\times\frac{4\times 6^3}{12}} = 0.0853\text{ in.}$$

With respect to axis 2-2,

$$d = \frac{100 \times 48^3}{3 \times 1{,}200{,}000 \times \frac{6 \times 4^3}{12}} = 0.096 \text{ in.}$$

$$\text{Resultant } d = \sqrt{0.0853^2 + 0.096^2} = 0.128 \text{ in.}$$

$$\tan \theta = \frac{0.096}{0.0853} = 1.125$$

$$\theta = 48°20' \text{ with the vertical}$$

Problems

308. A wooden beam 10 by 12 in. and 14 ft. long is supported at the ends. The beam carries a total uniform load of 9,000 lb. and a concentrated load at the middle of the span of 3,600 lb. What will be the maximum bending stress, shearing stress, and deflection? Are the values within safe limits? $E = 1{,}200{,}000$.

309. Using the allowable stresses of Table XVII in the Appendix, what safe uniformly distributed load will a 6- by 10-in. actual southern pine beam carry if the span is 10 ft.? What would be the load per linear foot if the span were 5 ft.? Assume the best location, and use select grade.

310. For what length of beam will the total uniform load be determined by shear only, and for what length by bending? Assume a southern pine beam select in the best location and 8- by 10-in. actual cross section.

311. A wooden beam 6- by 10-in. has a span of 12 ft. What will be the respective deflections with the 6-in. side and then the 10-in. side placed vertically? How is the deflection related to the dimensions of the beam? $E = 1{,}200{,}000$.

312. On the basis of the allowable deflection, what is the proper spacing from center to center of 10 in. 30-lb. I beams, 20 ft. span, to support a uniform floor load of 120 lb. per sq. ft. of floor surface?

313. If 7-in. 20-lb. I beams are spaced 6 ft. from center to center to support a floor load of 110 lb. per sq. ft. of surface, what is the limiting span, maximum deflection being allowed. What are the maximum bending and shearing stresses?

314. An 18-in. 60-lb. I beam carries a uniformly distributed load of 48,000 lb. on a simple span of 24 ft. Compute the maximum bending stress, shearing stress, and deflection.

315. A southern pine beam select, 4 by 8 in. and 7 ft. long, is used as a cantilever to support an end load of 900 lb. What are the maximum bending stress, shearing stress, and deflection?

316. A floor carrying a load of 140 lb. per sq. ft. is supported by 12-in. 35-lb. I beams spaced 6 ft. from center to center. The span of the beams is 21 ft. Does the deflection exceed $\frac{1}{360}$ of the span. What are the maximum bending and shearing stresses?

317. A steel shaft 10 in. in diameter resting in bearings 42 in. apart carries a flywheel weighing 20 tons midway between the bearings. Treating the shaft as a simple beam, find the maximum values of bending stress, shearing stress, and deflection.

318. If the beams in Prob. 316 are supported by a crossbeam 18 ft. long, what will be the size of the crossbeam? What will be the deflection? The loads transferred to the crossbeam are considered as concentrated loads at the third points. This floor represents an interior panel.

319. In Prob. 315, an additional horizontal force of 100 lb. is applied at the end of the beam and at right angles to it. What will be the amount and the direction of the resultant deflection?

320. A 4-in. steel shaft 10 ft. long and supported at the ends carries a pulley, weighing 200 lb., midway between the supports. There is a belt pull in a horizontal direction of 500 lb. acting on the pulley. What will be the amount and the direction of the resultant deflection?

CHAPTER XV

COMBINED STRESSES—SHORT COLUMNS

166. Direct and Bending Stresses.—It often happens that a beam or structure is carrying a load producing direct tension or compression and at the same time a load producing bending fiber stresses. Take, as an example, a vertical cantilever beam, *AB* in Fig. 379, fixed at *B* and supporting a load of 1,000 lb. This load causes uniform compressive fiber stresses in *AB*. If at the same time a horizontal pressure of 400 lb. acts at *D*, a bending moment is set up in the structure, which causes compression in the fibers of the beam on the side *C* and tension in the fibers on the side *D*. The result is a compression on the side *C* equal to the sum of the two compressions and a stress on the side *D* equal to the difference of the tensile and compressive stresses. Whether the resultant stress on the side *D* is tension or compression depends upon which of the two original stresses is the greater.

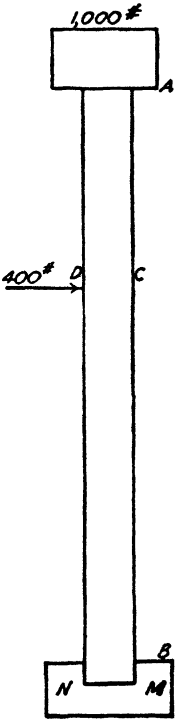


FIG. 379.

Suppose that in Fig. 379 the beam is 4 by 4 in. The fiber stress per square inch is

$$\frac{1{,}000}{4 \times 4} = 62.5 \text{ lb.}$$

Since it is uniformly distributed over the entire area, the stress will be represented as in Fig. 380.

Now, when a force produces bending moment, the maximum stress is in the outer fibers, decreasing to zero at the neutral axis. These stresses may be shown as in Fig. 381. The resultant maximum compressive stress on the right-hand fibers of the base is formed by adding *DC* and *GH*. That at *N* is formed by subtracting *EF* from *BA* or *BA* from *EF*, depending on which is the greater numerically. If *BA* and *EF* are of equal length, the resultant stress is zero.

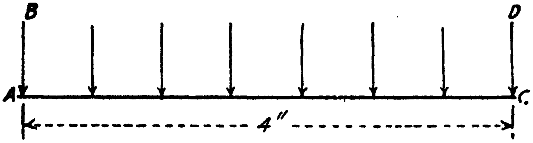


FIG. 380.

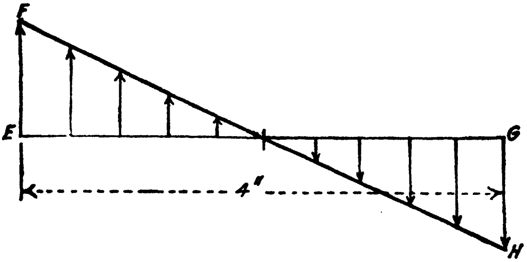


FIG. 381.

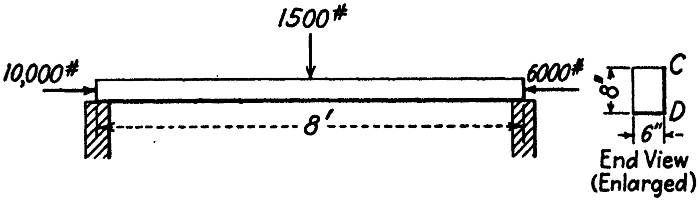


FIG. 382.

Example.—Find the maximum fiber stress developed in a simple wooden beam 6 by 8 in., having a span of 8 ft. and carrying a concentrated load of 1,500 lb. at the center and an end compression load of 10,000 lb. (Fig. 382).

Solution.—Since the cross-sectional area of the beam is 48 sq. in., the compressive fiber stress per square inch developed by the 10,000-lb. load is

$$s_c = \frac{10,000}{48} = 208.3 \text{ lb.}$$

This stress is shown in Fig. 383 where *EF* represents the depth *CD* of the beam in Fig. 382. For the fiber stress due to 1,500 lb., we have

$$M = \frac{pI}{c}$$

where p is the fiber stress, I the moment of inertia, and c the distance from the neutral axis to the extreme outer fiber.

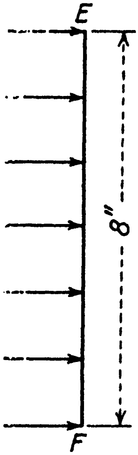


FIG. 383.

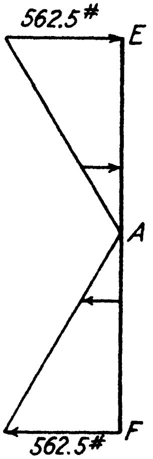


FIG. 384.

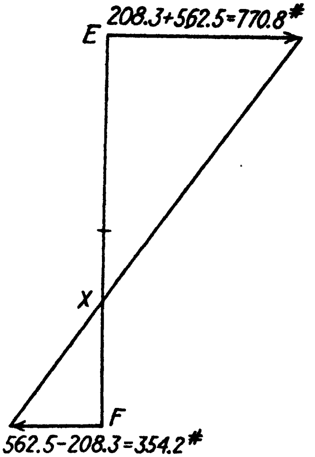


FIG. 385.

But, B.M. $= \frac{1}{4}Wl =$

$$\frac{1{,}500 \times 8 \times 12}{4} = 36{,}000 \text{ in.-lb.}$$

$$I = \frac{1}{12}bh^3 = \frac{6 \times 8 \times 8 \times 8}{12} = 256$$

and

$$c = 4 \text{ in.}$$

Then

$$\frac{256p}{4} = 36{,}000$$

$$p = 562.5 \text{ lb./sq. in.}$$

This force is shown in Fig. 384, where *EF* represents *CD*, the height of the beam.

Figure 385 shows the combined fiber stresses acting along *CD* of Fig. 382. At *E* the resultant stress is the sum of the two stresses, or 770.8 lb. At *F* the stress is 562.5 − 208.3 = 354.2 lb./sq. in. tension. Then the maximum fiber stress is 770.8 lb./sq. in.

167. Derivation.—In the formula for bending moment, solve for p the fiber stress.

$$p = \frac{Mc}{I}$$

Also, $s_c = P/A$, where P is the total end load, A the area of the cross sections, and s_c the uniform fiber stress due to the compressive load.

Then the maximum fiber stresses developed in the beam, if represented by f, are

$$f = s_c \pm p = \frac{P}{A} \pm \frac{Mc}{I} \qquad (61)$$

If the end loads are compression forces, the plus sign will give the maximum compressive fiber stress for designing. The minus sign gives the fiber stress on the opposite side of the beam, which may be tension or compression.

If P/A is greater than Mc/I, the resultant stress is compression. If P/A is less than Mc/I, the resultant stress is tension.

In the case of a steel or wooden beam, not much attention need be paid to the tensile stress. But with cast iron for which a safe working tensile stress is only 3,000 lb. per sq. in., whereas the safe working compressive stress is 15,000 lb. per sq. in., both fiber stresses must be carefully considered. For example, if $Mc/I - P/A$ should exceed 3,000 lb., although the fiber stress is safe on the compression side the designer must select the beam to withstand the tension fiber stress by decreasing the load or increasing the size of the beam.

In the case of brickwork or stone walls, no tensile strain is supposed to be thrown into the structure. Consequently, in such cases, Mc/I must never exceed P/A, for a force would be thus indicated that would tend to open up the joints, especially toward the top of the structure where the weight of the material is not sufficient to offset the tensile force.

If the problem is to find the beam for a certain loading, when the allowable fiber stress is given, the solution is quite complicated. However, the designer could handle the problem by the so-called "cut and try" method—that is, he could assume a certain sized beam and substitute the values in the right-hand side of Eq. (61). This value should come as near f, a safe working stress, as possible, but should never exceed it.

Example 1.—Find the size of a square wooden beam, of span 12 ft., to carry a uniformly distributed load of 4,000 lb. and end compression loads of 4,000 lb.

Solution.—Let h be the side of the square section of the beam. $M = \frac{1}{8} \times 4{,}000 \times 12 \times 12 = 72{,}000$ in.-lb. $c = h/2$, $I = \frac{1}{12}h^4$, $P = 4{,}000$, $A = h^2$, and $f = 1{,}000$ lb.
Substituting in (61),

$$1{,}000 = \frac{4{,}000}{h^2} + \frac{72{,}000}{h^4/12} \times \frac{h}{2}$$

$$1 = \frac{4}{h^2} + \frac{432}{h^3}, \qquad \text{or } h^3 - 4h - 432 = 0$$

We shall not try to find the exact value of h that makes the equation zero. The object will be to find two consecutive integral values of h, which when substituted in $h^3 - 4h - 432$ will give results opposite in sign. Then the larger value of h is the one to use.

Try $h = 7$.

$$7 \times 7 \times 7 - 4 \times 7 - 432 = -117$$

Let $h = 8$.

$$8 \times 8 \times 8 - 4 \times 8 - 432 = +48$$

Since $h^3 - 4h - 432$ changes from minus to plus when $h = 7$ and $h = 8$, the dimensions for the square timber are 8 by 8 in.

In the problem solved, the load producing bending was taken at the center of the beam. If this force is not at the center, first calculate the end reactions. Then either reaction multiplied by its distance in inches from the load will give the value of M to substitute in Eq. (61).

In the case of a post or column fastened at one end with a load on the top, and a side force causing bending, we have a cantilever effect. The maximum fiber stresses occur at the bottom, and M is simply the side force multiplied by its distance in inches from the bottom of the column or strut.

Example 2.—A wooden post 8 in. square and 6 ft. high carries a load of 12,800 lb. and is acted on to the right by a force of 300 lb. applied 2 ft. from the top (Fig. 386). Find the fiber stresses in AB and CD.

Solution.—$P = 12{,}800$ lb., $A = 8 \times 8 = 64$ sq. in., $c = 4$ in.

$$I = \frac{1}{12}bh^3 = \frac{8 \times 8 \times 8 \times 8}{12} = 341.3$$

The moment M at the bottom, where the danger section is located, is

$$M = 300 \times 4 \times 12 = 14{,}400 \text{ in.-lb.}$$

Substituting in Eq. (61),

$$f = \frac{12{,}800}{64} \pm \frac{14{,}400 \times 4}{341.3} = 200 \pm 169 \text{ approximately}$$

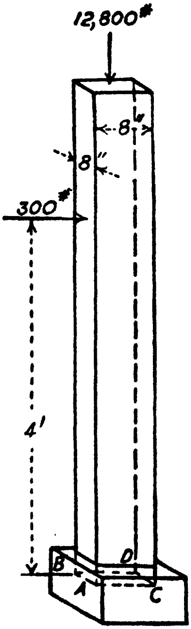


Fig. 386.

The 300-lb. force causes tension in AB and compression in CD.

Then

$$f = 200 + 169 = 369 \text{ lb./sq. in. compression in } CD$$

Also,

$$f = 200 - 169 = 31 \text{ lb./sq. in. compression in } AB$$

Problems

321. A wooden beam with a 12 ft. span carries a load of 80 lb. per ft. and sustains an axial compression of 1,500 lb. If a stress of 1,200 lb. per sq. in. is allowed, what should be the dimensions of a square beam?

322. What should be the size of a square wooden beam for a simple span of 10 ft. to support a load of 800 lb. at the center and an axial compression of 4,200 lb.? Use stress as in Prob. 321.

323. A 10-in. 40-lb. I beam 9 ft. long carries a uniform load of 1,200 lb. in addition to its own weight and also resists an axial force of 48,000 lb. What is the factor of safety?

324. A wooden beam with a 9 ft. span and 8 by 10 in. in cross section carries a total uniform load of 3,600 lb. and an axial compression of 28,000 lb. Compute the maximum stress.

325. A simple beam 8 in. wide and 12 ft. long carries a uniform load of 400 lb. per ft. and a compression load of 24,000 lb. What is a proper depth for the beam?

326. A cantilever 4 by 6 in. and 4 ft. long carries a load of 400 lb. at the free end. If the stress shall not exceed 1,200 lb. per sq. in., what axial compression may be applied?

327. A simple beam 6 by 8 in. in cross section and 12 ft. long is to resist an end compression of 20,000 lb. What total uniform load may be added so that the maximum stress shall be equal to 1,200 lb. per sq. in.?

168. Short Columns.—When the length of a structural member is not greater than ten times the least dimension of the cross section, it is called a *short column* or *block*. If axial loads are applied to the end of the block, it is considered a tension or compression member. No buckling effect is considered. For example, a block 4 by 8 in. and 3 ft. long is a short column, for the length 36 in. is less than 10 × 4 = 40 in.

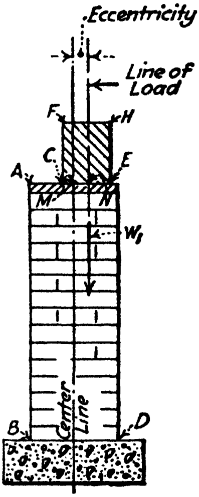


Fig. 387.

When forces are applied to the ends of a block, if the center of the bearing area is not axial the force is said to be *eccentric*.

Take, for example, a brick pier resting on a concrete foundation, carrying a beam that is not placed in the center (Fig. 387). *CFHE* is the end view of this beam. Let W_1 be the weight carried by the beam, acting through its center line *N* to the pier. Let *a* be the length *MN* from the center of the pier to the center line of the load. That is, the amount by which the load is eccentric is *a*. It is proved in mechanics that when a load W_1 is carried off center the crushing effect on the pier or column is the same as if the load were central, and in addition a bending moment is set up at the base of the pier around the neutral axis perpendicular to the edge *BD*. The moment of the couple W_1 is aW_1. Then from the equation,

$$\text{B.M.} = p\frac{I}{c},$$

$$aW_1 = p\frac{I}{c}, \qquad \text{or } p = \frac{acW_1}{I}$$

This value is the fiber stress in the extreme outer fibers of the pier along the edges vertical to the edge *BD* at *B* and *D*. Figure 388 shows the relations. *PR* is the neutral axis, and p_1 is the compressive stress in the fiber *TD* and the tensile stress in *SB*. W_2 is the total load carried at the bottom of the pier, consisting of the load W_1 and the weight of the pier. The compressive fiber stress p_2 due to this load is

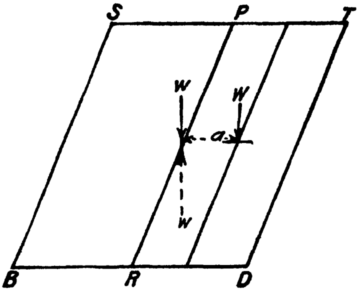


Fig. 388.—Cross section of base of pier.

$$p_2 = \frac{W_2}{A}$$

where *A* is the area of the cross section of the pier. Then the stresses on *TD* and *SB*, respectively, are

$$f = p_2 \pm p_1 = \frac{W_2}{A} \pm \frac{W_1ac}{I} \tag{62}$$

Example 1.—Suppose that a beam, one support of which is shown in Fig. 387, carries a uniformly distributed load of 22,000 lb. The pier is 12 in. square and 8 ft. high. Find the maximum fiber stress.

Solution.—When the end load of the beam is central on the pier, the uniformly distributed load on the base of the pier consists of one-half the total load, or

$$\frac{22{,}000}{2} = 11{,}000 \text{ lb.}$$

and the weight of the pier. If brick weighs 120 lb. per cu. ft. the total weight is

$$\frac{12 \times 12 \times 8 \times 120}{144} = 960 \text{ lb.}$$

$$\therefore W_2 = 11{,}000 + 960 = 11{,}960 \text{ lb.}$$

Then

$$p_2 = \frac{11{,}960}{12 \times 12} = 83 \text{ lb./sq. in.}$$

If this load is off center, say 3 in., very different stresses exist. Here $a = 3$, and $c = 6$.

$$I = \frac{1}{12}bh^3 = \frac{12 \times 12 \times 12 \times 12}{12} = 1{,}728, \qquad W = 11{,}000$$

$$\therefore p_1 = \frac{acW}{I} = \frac{3 \times 6 \times 11{,}000}{1{,}728} = 115 \text{ lb. approximately}$$

Then the maximum compression stress on DT (Fig. 388) is $f = 83 + 115 = 198$ lb./sq. in., and in BS it is $f = 83 - 115 = -32$ lb./sq.in. This shows that the left-hand fibers have been thrown into tension, which is not safe for stonework or brickwork. A safe working compression stress for brickwork laid with lime mortar is 110 lb. and for Portland cement 200 lb. For rubble stonework laid in lime mortar, a safe stress is 75 lb. per sq. in., or 140 lb. when Portland cement is used. In any case, the loading above is not safe. The stresses may be brought within safe limits by reducing a to less than 2 in. or increasing A of the cross section of the pier or both at the same time.

Example 2.—Let $a = 1\frac{1}{2}$ in. and $A = 15 \times 15 = 225$ sq. in. Then

$$W_1 = 11{,}000 + \frac{15 \times 15}{144} \times 8 \times 120 = 12{,}500$$

Substituting in Eq. (62),

$$f = \frac{12{,}500}{225} \pm \frac{11{,}000 \times 1\frac{1}{2} \times 7\frac{1}{2}}{\frac{1}{12} \times 15 \times 15 \times 15 \times 15} = 55.6 \pm 29.3$$

$f = 85$ lb./sq. in. compression on one side and 26.3 lb./sq. in. compression on the other.

169. Eccentric Loads on Short Columns.—In brick or cement work where little or no tension is desirable, the tensile stresses must not be allowed to become greater than the compressive stresses. This requirement can be governed entirely by observing the following rule:

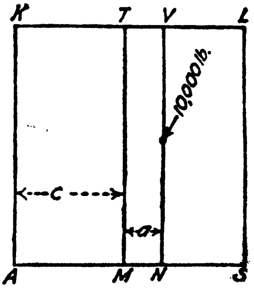


FIG. 389.

In Fig. 389, $AKLS$ represents the top view of a block on which a load W_1 acts. The center line of the load is NV and the neutral axis MT. The rule to follow in the case of rectangular columns is that *if the load is placed so that the distance a is less than ⅙AS, the side of the rectangle, then the combined stresses at AK cannot be tension. If the* *load were placed between TM and AK and less than $\frac{1}{6}AS$ from TM, the stresses over the entire section would be compressive—i.e., if the load is anywhere in the middle third of the section of a rectangular column, there is no resultant tensile stress.*

The proof is as follows:

Now p_2 must not be greater than p_1. Neglecting the weight of the column, *i.e.*, making $W_1 = W_2$ and $p_1 = p_2$ of (62), we have

$$\frac{W_1}{A} = \frac{W_1 ac}{\frac{1}{12}bh^3}$$

But $A = bh$ and $c = h/2$. Substituting these values,

$$\frac{1}{bh} = \frac{a(h/2)}{bh^3/12}$$

$$\therefore a = \frac{h}{6}$$

Since the condition is true for $W_1 = W_2$, it will necessarily be true when $W_2 > W_1$, since it increases p_2 and therefore makes $p_2 - p_1$ positive.

Example 1.—Suppose the column of Fig. 389 is 12 in. square and 10 ft. high. Then $AS = AK = 12$ in. Since the height 120 in. is only ten times the lateral dimension 12 in., the pier may be treated as a short column. Let $W_1 = 10{,}000$ lb., acting along the line *VN*. Now, $\frac{1}{6}$ *AS*, or 12 in., gives 2 in. To illustrate the working of the above rule, the stresses will be found for $a = 1\frac{1}{2}$ in., $a = 2\frac{1}{2}$ in., and $a = 2$ in. In the first case, the resultant stresses will be compressive, whereas in the second case the results will show compression on *LS* and tension in *AK*. In the third case, we shall find compression on *LS* with zero or no stress in *AK*. The load W_1 will cause a uniform compressive stress on the fibers of the beam equal to the load divided by the area of the cross section. Let p_1 represent this stress. Then

$$p_1 = \frac{10{,}000}{12 \times 12} = 69.5 \text{ lb./sq. in.}$$

The stresses are shown in Fig. 390.

Now a load of 10,000 lb. applied along the line *NV* causes a turning effect about the neutral axis *MT*. This results in compressive

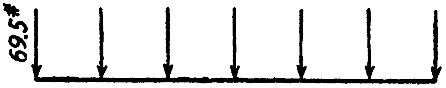


FIG. 390.

stresses on the fibers to the right of *MT* and tension in the fibers to the left of *MT*. From the equation

$$\text{B.M.} = \frac{p_2 I}{c}$$

I is the moment of inertia of the rectangle about *MT*, p_2 is the fiber stress in the extreme outer fibers, and *c* is *MS* or *AM*.

$$\text{B.M.} = 10{,}000 \times MN \times 10{,}000 \times 1\tfrac{3}{4} = 17{,}500 \text{ in.-lb.}$$

$$I = \frac{bh^3}{12} = \frac{1}{12} \times 12 \times 12 \times 12 \times 12 = 1{,}728, \qquad \text{and } c = 6$$

$$\therefore p_2 \times \frac{1{,}728}{6} = 17{,}500$$

$$p_2 = \frac{17{,}500}{288} = 60.8 \text{ lb.}$$

These stresses are shown in Fig. 391. They form triangles because there is no stress at *M*, the neutral axis, and these forces increase uniformly to $p_2 = 60.8$ at the outermost fibers.

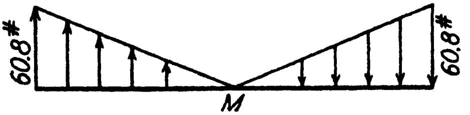


FIG. 391.

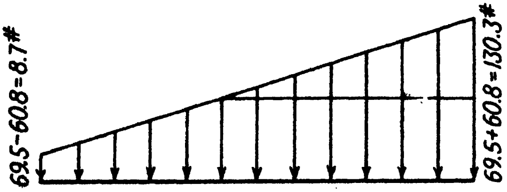


FIG. 392.

Since both sets of stresses act at the same time, the resultant effect on the fibers of the section *AKLS* is a compressive stress equal to $60.8 + 69.5 = 130.3$ and decreasing to $69.5 - 60.8 = 8.7$ lb. The bending stresses on *MAKT* act opposite to the compressive stresses, but they are always less. Figure 392 shows the combination. Now, if $a = 2\frac{1}{2}$ in., then B.M. = 10 000 × $2\frac{1}{2}$ = 25,000 in.-lb. I and c remain the same. Then

$$25{,}000 = \frac{p_2 \times 1{,}728}{6}$$

$$p_2 = \frac{25{,}000}{288} = 86.8 \text{ lb./sq. in.}$$

Combining p_1 and p_2, we have a compressive stress at LS of Fig. 389 equal to 86.8 + 69.5 = 156.3 lb. But at AK the upward, or tensile, pull due to bending of 86.8 is greater than the direct compressive stress of 69.5. We then have a tensile stress of 17.3 lb. along AK. Figure 393 shows the resultant stresses.

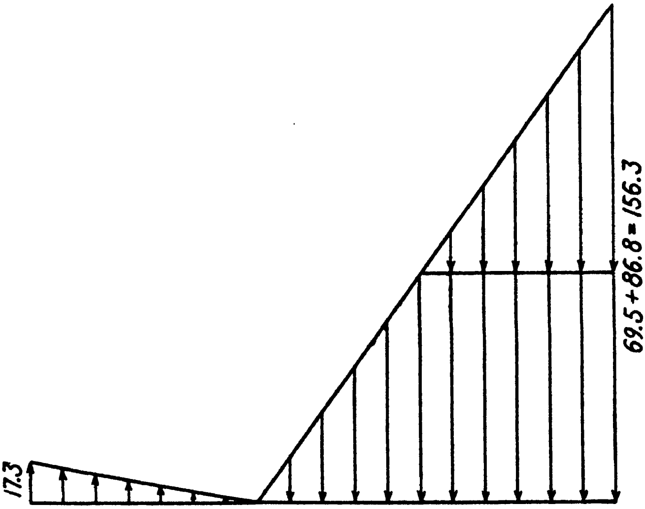


FIG. 393.

Now let $a = 2$. Then,

$$10{,}000 \times 2 = p_2 \times \frac{1{,}728}{6}$$

$$p_2 = \frac{20{,}000}{288} = 69.5$$

Figure 394 shows the combined stresses. When the load is at the middle third, the stresses on AK is just equal to zero.

This problem verifies the rule that, when a load is placed on a rectangular pier, if the center of the bearing area is inside the middle third of the rectangle all fibers are in compression. If necessary to place the load outside the middle third, then the resultant stress on one side of the pier is tension and the other compression. In case the load must be placed so that tensile stresses result, then the pier must be of a material to stand tension.

170. Location of the True Neutral Axis.—The neutral axis for members subjected to both bending and direct stress is not identical with the gravity axis as is the case when there is bending stress only. In all cases, it is understood that the true neutral axis is the axis where the stress is zero.

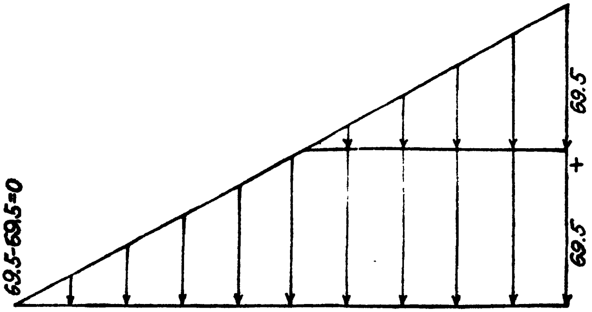


FIG. 394.

In the example in Art. 166, the stress in the top of the beam is 770.8 lb. per sq. in. compression and in the bottom it is 354.2 lb. per sq. in. tension. Since the stresses are proportional to the respective distances from the true neutral axis, the following equations may be written:

$$\frac{EX}{XF} = \frac{770.8}{354.2}$$

Also,

$$EX + XF = 8 \text{ in.}$$

Then

$$\frac{EX}{8 - EX} = \frac{770.8}{354.2}, \qquad 354.2EX = 8 \times 770.8 - 770.8EX$$

$$354.2EX + 770.8EX = 6{,}166.8$$

$$EX = \frac{6{,}166.8}{1{,}125} = 5.53 \text{ in.}$$

The neutral axis is 5.53 in. below the top of the beam; as has been suggested before, it is not identical with the gravity axis when there are both bending and direct stresses.

In Example 2 of Art. 167, the stresses in the extreme fibers are both compressive. In this case, there can be no zero stress within the beam. In Fig. 386, the true neutral axis lies to the left of the section. Let x be the distance from the neutral axis to AB. A proportion may be written for the two stresses as follows:

$$\frac{x}{x+8} = \frac{\text{stress in } AB}{\text{stress in } CD}, \qquad \text{or } \frac{x}{x+8} = \frac{31}{369}$$

Solving, we have

$$\begin{aligned} 369x &= 31x + 248 \\ 338x &= 248 \\ x &= \tfrac{248}{338} = 0.734 \text{ in.} \end{aligned}$$

The true neutral axis is 0.734 in. to the left of the section, whereas, if bending alone were considered, it would lie midway between AB and CD.

Problems

328. What is the position of the true neutral axis in Example 1 of Art. 168? Sketch a diagram showing the stress vectors and indicate point of zero stress.

329. In Prob. 325, Art. 167, determine the position of the axis of zero stress. Illustrate by means of a diagram.

171. Circular Columns.—*In the case of a circular pier or column, if the load is inside the middle fourth, then all fibers are in compression.*

The proof of the rule for a circular pier is left to the student.

Example 1.—A round solid steel shaft 8 in. in diameter stands vertical and carries an eccentric load P, as in Fig. 395. Now, one-fourth of 4 = 1 in. Then if the center of the load is placed anywhere within the circle AB of radius 1 in., the resultant stresses are compression.

If the student should solve for the stresses in a solid circular pier, the only change in the formula as used is in the moment of inertia. For a circle the moment of inertia $I = \frac{1}{4}\pi r^4$, where $\pi = 3.1416$ and r is the radius.

For example, in the circle of Fig. 395, $r = 4$ and

$$I = \tfrac{1}{4} \times 3.1416 \times 4 \times 4 \times 4 \times 4 = 201 \text{ in.}^4$$

Suppose the pier carries a load of 20,000 lb. placed ¾ in. from the center. Then $a = \frac{3}{4}$, $c = 4$ in., the distance from the center to the extreme fiber.

$$\text{B.M.} = 20{,}000 \times \tfrac{3}{4} = 15{,}000 \text{ in.-lb.}$$

Then, since $\text{B.M.} = \frac{p_1 I}{c}$,

$$15{,}000 = \frac{p_1 \times 201}{4}$$

$$p_1 = \frac{4 \times 15{,}000}{201} = 300 \text{ lb.}$$

In Fig. 395*a*, the center of the bearing area is *H*. The radius *OM* is drawn through *H*. The diameter *CD* is drawn perpendicular to *OM*. *CD* is the neutral axis, or line of no stress. The fibers of the column below *CDM* are in compression which increases from zero at *O* to 300 lb. per sq. in. at *M*, whereas below *CDN* the fibers are in tension, the stress at *N* being 300 lb. per sq. in. The load at *B* produces a compressive stress over the entire circle just as if it were at *O*. To find the stress per square inch, we divide the load by the area or

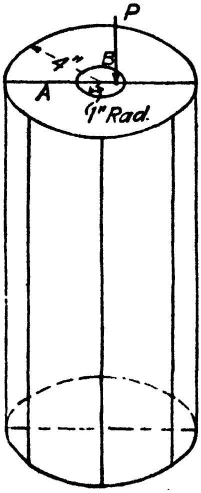


FIG. 395.

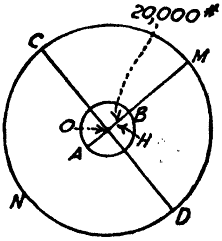


FIG. 395*a*.

$$p_2 = \frac{20{,}000}{\pi r^2} = \frac{20{,}000}{\pi \times 4 \times 4} = 400 \text{ lb.}$$

Then at *M* the resultant stress is 300 + 400 = 700 lb.

At N the resultant stress is $400 - 300 = 100$; *i.e.*, the fibers at N are in compression. For safe design, the added stress at the bottom directly under the point M, due to the weight of the pier itself, must be found. If W is the weight, then the stress per square inch on the lower base of the column is

$$p_2 = \frac{W}{\pi r^2}$$

Then the total stress for the design is $300 + 400 + (W/\pi r^2)$.

172. Eccentric Riveted Connections.—Eccentric loading is frequently found in riveted connections. Whenever possible, eccentric loads are to be avoided by constructing the joint so that the resultant force to be resisted by the rivets will pass through the center of gravity of the group of rivets. If this is not possible and the force must be eccentric with respect to the center of the group of rivets, then, as before, its effect will be resolved into that of an axial force and a couple. Thus, in Fig. 396, the pull of 12,000 lb. as actually applied is equivalent to a pull of 12,000 lb. acting through the middle rivet and a moment of 12,000 × 1.5 = 18,000 in.-lb. There will be a direct stress

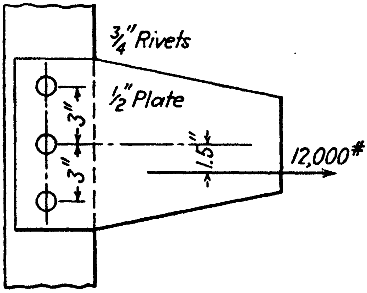


Fig. 396.

$$s_s = \frac{12{,}000}{3 \times 0.44} = 8{,}523 \text{ lb./sq. in.}$$

The middle rivet will have no stress due to rotation, but the two outside rivets will be equally stressed since they are at equal distances from the center of the group. Since resisting moment must be equal to the moment of the couple,

$$2 \times 0.44 \times s_s \times 3 = 18{,}000$$

$$s_s = \frac{3{,}000}{0.44} = 6{,}818 \text{ lb./sq. in.}$$

The direct stress is parallel to the pull. The stress due to rotation is perpendicular to the line of rivets and for this arrangement of rivets is parallel to the pull. Therefore, the resultant stress on the outer rivets will be

$$s_s = 8{,}523 + 6{,}818 = 15{,}341 \text{ lb./sq. in. } Ans.$$

On the middle rivet, the stress will be the direct value only, or

$$s_s = 6{,}818 \text{ lb./sq. in. } Ans.$$

173. The Middle Third in Dams.—The rule stated in Art. 169 is called the *principle of the middle third.* This principle is carefully observed in problems relating to loads carried by masonry and brick piers or to loads on wall footings. It is also of prime importance in the design of masonry arches, retaining walls, and dams. The line of action of the resultant is required to pass within the middle third of any section of an arch and within the middle third of the base for the dam and the retaining wall.

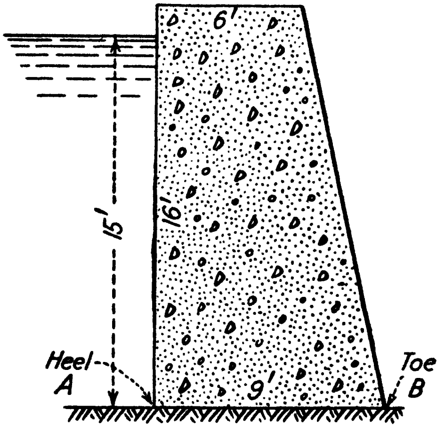


Fig. 397.

Example.—A dam shown in cross section in Fig. 397 impounds water in a reservoir. The water rises to within 1 ft. of the top of the dam. What will be the stresses at the heel and at the toe of the dam? Water weighs 62.5 lb. per cu. ft. Concrete weighs 150 lb. per cu. ft.

Solution.—In analyzing the dam, a section 1 foot in length is taken as the unit. This is similar to the repeating sections in the analysis of riveted joints.

The weight of dam per linear foot is

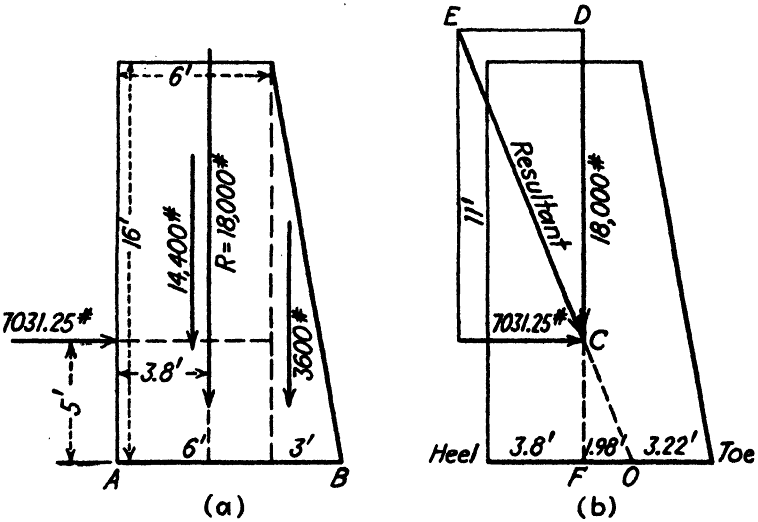
$$W = (6 \times 16) + \frac{(3 \times 16)}{2}150 = 18{,}000 \text{ lb.}$$

The line of action of the resultant weight is found by taking moments about the heel A. Then

$$18{,}000x = 14{,}400 \times 3 + 3{,}600 \times 7$$

$$x = \frac{43{,}200 + 25{,}200}{18{,}000} = 3.8 \text{ ft.}$$

The weight of the dam acts at a distance of 3.8 ft. from the heel.



Figs. 397*a*, *b*.

The water pressure acts at one-third of the depth, or 5 ft. above the base. The amount of the water pressure per foot of length is

$$P = \frac{62.5 \times 15 \times 15}{2} = 7{,}031.25 \text{ lb.}$$

Figures 397*a*, *b* show the forces acting on the dam.

The dam may be considered as a vertical cantilever beam in which there are a direct stress due to the weight of the masonry and bending stresses due to the water pressure and to the weight of masonry as an eccentric load.

The direct stress at A is

$$s_A = \frac{18{,}000}{9 \times 1} = 2{,}000 \text{ lb./sq. ft. compression}$$

The stress due to bending at A is

$$s_A = \frac{(18{,}000 \times 0.7)4.5}{1 \times (9^3/12)} - \frac{(7{,}031.25 \times 5)4.5}{1 \times (9^3/12)}$$
$$s_A = 933.3 - 2{,}604.1 = -1{,}670.8 \text{ lb./sq. ft. tension}$$

The total stress at A is

$$\begin{aligned} s_A &= 2{,}000 - 1{,}670.8 = 329.2 \text{ lb./sq. ft. compression} \\ &= 2.28 \text{ lb./sq. in. compression} \end{aligned}$$

The direct stress at B, is the same as at A, or

$$s_B = 2{,}000 \text{ lb./sq. ft. compression}$$

The water pressure will cause a compressive bending stress at B, and the bending stress due to eccentric load will be tensile. The stress due to bending at B is

$$s_B = -933.3 + 2{,}604.1 = 1{,}670.8 \text{ lb./sq. ft.}$$

The total stress at B is

$$\begin{aligned} s_B &= 2{,}000 + 1{,}670.8 = 3{,}670.8 \text{ lb./sq. ft} \\ &= 25.5 \text{ lb./sq. in.} \end{aligned}$$

Since both values of stress are compressive, it is evident that the resultant of the weight and the water pressure must intersect the base within the middle third. This point of intersection will now be located. In Fig. (397*b*), it can be seen that the triangles *CDE* and *CFO* are similar. Then

$$\frac{FO}{CF} = \frac{ED}{CD}$$

Or

$$\frac{FO}{5} = \frac{7{,}031.25}{18{,}000}$$
$$FO = 1.98 \text{ ft.}$$

The resultant cuts the base at a point 3.8 + 1.98 = 5.78 ft. from the heel. This is just within the middle third but close to the downstream third point.

174. Shear Due to Tension or Compression.—If an axial force is applied to a member, the stress, which might be either tension or compression, is uniformly distributed over the normal cross section. There is no component of force parallel to the section, and consequently there is no shearing stress produced on the section.

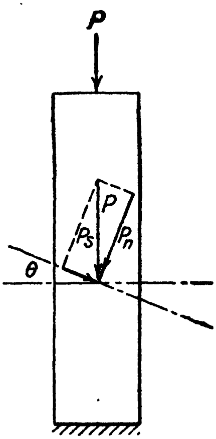


FIG. 398.

If a section making some angle Θ with the normal section is chosen as shown in Fig. 398, then the axial force may be resolved into two components, one at right angles to the section, called P_n, and the other parallel to the section, called P_s. Then

$$P_n = P \cos \Theta, \qquad P_s = P \sin \Theta$$

The area resisting these forces will be the area of the inclined section and equal to $A/\cos\ \Theta$, where A is the area of the normal section. Then

$$s_n = \frac{P_n}{A_n} = \frac{P \cos \Theta}{A/\cos \Theta} = \frac{P \cos^2 \Theta}{A} \tag{63}$$

This value is a maximum when the angle $\Theta = 0°$. That is, it is seen that the maximum tension or compression will take place on the normal cross section. Also,

$$s_s = \frac{P_s}{A_s} = \frac{P \sin \Theta}{A/\cos \Theta} = \frac{P}{A} \sin \Theta \cos \Theta = \frac{P}{A}\frac{\sin 2\Theta}{2} \tag{64}$$

The value of sin 2Θ is largest when $\Theta = 45°$, and therefore the value of s_s is a maximum on a plane making an angle of 45° with the normal cross section.

Example 1.—A concrete post is 10 in. square. It carries a load of 20,000 lb. What are the stresses on a section making an angle of 30° with the normal section? What is the maximum value of s_s?

Solution.

$$P = 20{,}000 \text{ lb.}, \qquad A_n = \frac{10 \times 10}{\cos 30°} = \frac{100}{0.866} = 115.5 \text{ sq. ir}$$

$$P_n = 20{,}000 \cos 30° = 17{,}320 \text{ lb.}$$

$$s_n = \frac{17{,}320}{115.5} = 15{,}000 \text{ lb./sq. in. } \textit{Ans.}$$

$$P_s = 20{,}000 \sin 30° = 10{,}000 \text{ lb.}$$

$$s_s = \frac{10{,}000}{115.5} = 86.6 \text{ lb./sq. in. } \textit{Ans.}$$

$$s_{s_{\max}} = \frac{20{,}000}{2 \times 100} \sin (2 \times 45°) = 100 \text{ lb./sq. in.}$$

Example 2.—In Example 1, what load would cause a shearing stress of 250 lb. per sq. in.? What will be the normal stress?

Solution.

$$s_{s_{\max}} = 250 = \frac{P}{A}\frac{\sin (2\theta)}{2} = \frac{P}{100 \times 2} \sin 90°$$

Solving,

$$P = \frac{250 \times 200}{\sin 90°} = 50{,}000 \text{ lb. } \textit{Ans.}$$

$$s_n = \frac{50{,}000}{10 \times 10} = 500 \text{ lb./sq. in.}$$

Since the shearing is a maximum on the 45° plane, then on this plane

$$s_n = \frac{50{,}000}{100} \cos^2 45° = 500 \times 0.707^2 = 250 \text{ lb./sq. in.}$$

The normal and shearing stresses are equal on a plane making an angle of 45° with the normal cross section.

Example 3.—What is the greatest shearing stress in the riveted connection shown in Fig. 399?

Solution.—The load produces a direct shearing stress which is assumed to be the same for all rivets. A for a $\frac{7}{8}$-in. rivet = 0.6 sq. in.

$$s_s = \frac{10{,}000}{0.6 \times 8} = 2{,}080 \text{ lb./sq. in.}$$

The load also tends to rotate the plate in a clockwise direction. Since the combined resistance of all the rivets may be assumed to act at the center of gravity of the group of rivets, this point represents the neutral point and is taken as the center of moments.

The moment due to the load is

$$10{,}000 \times 10 = 100{,}000 \text{ in.-lb.}$$

For equilibrium, the resisting moment of the rivets must be equal to 100,000 in.-lb. The unit shearing stresses in the rivets will be proportional to their respective distances from the center of gravity of the group. These distances are 5.4 in. for rivets *a* and 3.35 in. for rivets *b*.

Let s_s represent the unit shearing stress due to moment in the outermost rivets *a*; then the unit stress in rivets *b* will be $(3.35/5.4)s_s$. Taking moments about the center of gravity of the group, we have

$$(4 \times 0.6 \times s_s \times 5.4) + \left(4 \times 0.6 \times \frac{3.35}{5.4}s_s \times 3.35\right) = 100{,}000$$

$$12.96s_s + 4.98s_s = 100{,}000$$

$$17.94s_s = 100{,}000$$

$$s_s = 5{,}580 \text{ lb./sq. in.}$$

The actual stress will be the resultant of 2,080 and 5,580. Before this resultant can be determined, the directions of the stresses must be known.

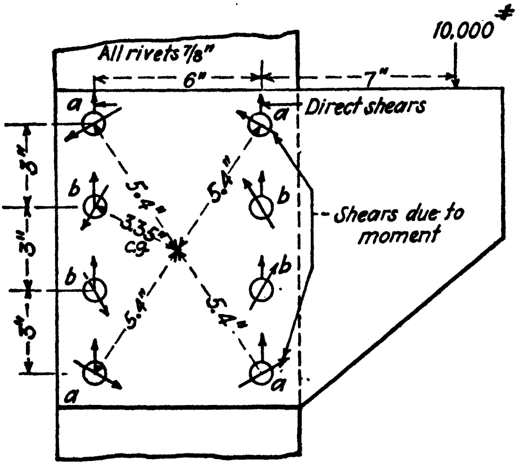


Fig. 399.

The 2,080 lb. is an upward force for each rivet. The 5,580 lb. for each rivet *a* is perpendicular to the radius arm of the rivet. The directions vary with the position of the rivet, as can be seen from the figure. By inspection, it is evident that both shears are upward at the upper right-hand rivet *a* and the resultant is greatest for this rivet.

The stress of 5,580 lb. makes an angle with the horizontal that is equal to the angle that the radius makes with the vertical. The tangent of this angle is 3/4.5 = 0.667, and the angle is 33°40′.

Resolving 5,580 lb. into its vertical and horizontal components, we have

$$F_y = 5{,}580 \sin 33°40' = 3{,}070 \text{ lb.}$$
$$F_x = 5{,}580 \cos 33°40' = 4{,}630 \text{ lb.}$$

The vertical and horizontal components of the resultant are

$$2{,}080 + 3{,}070 = 5{,}150 \text{ lb.}$$

and 4,630 lb., respectively.

The resultant

$$s_r = \sqrt{5{,}150^2 + 4{,}630^2} = 6{,}925 \text{ lb./sq. in.}$$

Problems

330. What would be the maximum and minimum stresses in the example in Art. 166, if additional loads of 1,500 lb. each were added at the quarter points of the beam?

331. What will be the stresses in the extreme fibers in a simple wooden beam 6 in. by 8 in. with a span of 8 ft. The beam carries a concentrated load of 2,000 lb. at the middle of the span and an end compression of 6,000 lb.

332. In a certain roof the loads supported by the rafters are as shown in Fig. 400. The rafters are 4- by 8-in. timbers. Find the maximum unit stress. If the walls are 16 ft. high, to what overturning are they subjected?

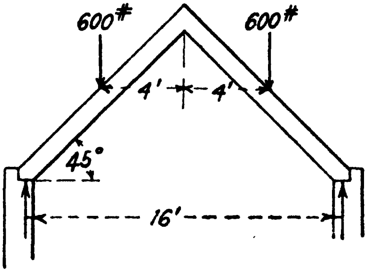


FIG. 400.

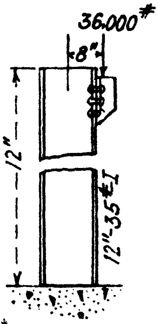


FIG. 401.

333. In a building, it was necessary to rivet a bracket to a column and support the load eccentrically as shown in Fig. 401. What are the maximum and minimum stresses in this I-beam column?

334. A city waterworks has a standpipe 80 ft. high and 25 ft. in diameter. The plates are $\frac{5}{8}$ in. thick at the bottom. The standpipe is exposed to a wind pressure of 30 lb. per sq. ft. of vertical projection. What are the stresses at the bottom, due to the water pressure? If the standpipe weighs 64,000 lb., what is the maximum stress at the base?

335. A symmetrical bridge pier is 6 ft. wide at the top and 10 ft. wide at the bottom. The reaction from a bridge weighing 400 tons falls 3 in. short of the center line of the base. What is the maximum stress at the base of the pier?

336. How thick is it necessary to build a concrete wall 20 ft. high if it is subjected to a wind pressure of 40 lb. per sq. ft.? The maximum stress is limited to 60 lb. per sq. in.

337. A factory smokestack is 120 ft. high. It has an outside diameter of 12 and an inside diameter of 10 ft. at the base, and an outside diameter of 9 ft. and an inside diameter of 7.5 ft. at the top. The stack tapers uniformly. In the design of the stack a wind pressure of 30 lb. per sq. ft. of vertical projection was provided for. What will be the maximum combined stress?

338. In order to support a platform, it was found necessary to provide a bracket riveted to a column as shown in Fig. 402. If s_s is limited to 8,000 lb. per sq. in., what is a proper rivet size?

339. A bracket riveted to a column supports a load due to a traveling crane. The connection is shown in Fig. 403. What is the shearing stress in the rivets?

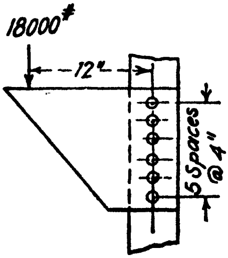


FIG. 402.

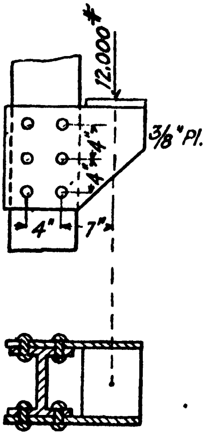


FIG. 403.

340. In Prob. 338, what may be the load if the stress in the outer rivets is not to exceed 10,000 lb. per sq. in.?

341. In Prob. 338, what should be the distance between rivets so that the resultant unit shearing stress in the outer rivets shall not exceed 10,000 lb. per sq. in.?

342. In Prob. 334, determine the location of the true neutral axis for the section. Illustrate by means of a diagram.

343. A dam 28 ft. high is 4 ft. thick at the top and 16 ft. thick at the base. The upstream face is vertical. If water rises to a depth of 27 ft., find where the resultant of the weight of the dam and the water pressure intersects the base. What are the stresses at the heel and the toe?

CHAPTER XVI

INDETERMINATE BEAMS—CONCRETE BEAMS

175. Discussion.—When a beam has one end fixed and the other end on a support, both ends fixed, or rests on three or more supports, the reactions and stresses cannot be found by $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$, and it is said to be "statically indeterminate." It is necessary to employ the expressions for deflection to furnish enough equations.

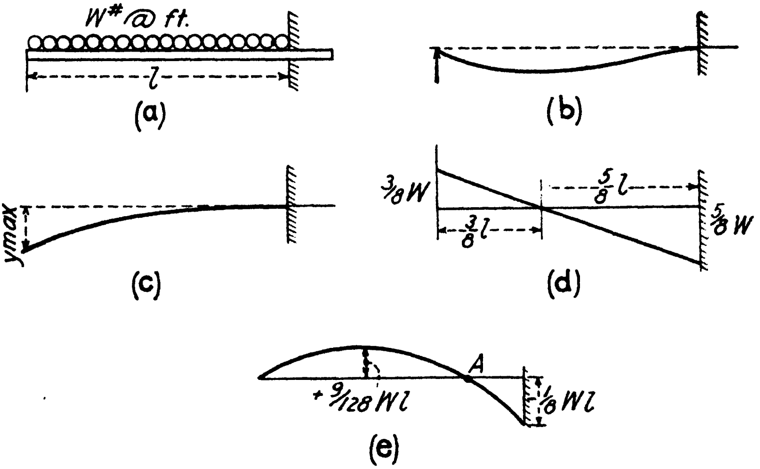


FIG. 404.—(*a*) Diagram of beam and load; (*b*) elastic curve; (*c*) elastic curve due to load alone; (*d*) shear diagram; (*c*) moment diagram.

176. Beam Fixed at One End and Supported at the Other.—Figure 404 shows a beam with the right end built into the wall and the left end on a support; it carries a uniformly distributed load. Figure 404*b* shows the elastic curve. If the support were removed, the beam would be a cantilever. Figure 404*c* shows the elastic curve due to load alone.

$$y_{\max} = \frac{1}{8}\frac{Wl^3}{EI}$$

The reaction of the support is a load R on the free end that deflects the end upward an equal distance or

$$y_{\max} = \frac{1}{3}\frac{Rl^3}{EI}$$

Then

$$\frac{1}{3}\frac{Rl^3}{EI} = \frac{1}{8}\frac{Wl^3}{EI}$$

$$R = \tfrac{3}{8}W$$

With R known, the shear and moment diagrams are constructed as shown in Figs. 404*d* and *e*. The shear diagram passes through zero at $\frac{3}{8}l$ and l. The bending moment at these points is $\frac{9}{128}Wl$ and $-\frac{1}{8}Wl$, respectively.

Since the maximum M is $-\frac{1}{8}Wl$, this beam is no stronger than a simple beam. At A of Fig. 404*e*, $M = 0$. This is called a "point of inflection." The only fiber stress is shear.

Example 1.—A balcony is supported by 6-in. 17.25-lb. I beams spaced 10 ft. on centers. The balcony projects 6 ft. from the wall and carries 200 lb. per sq. ft. of floor area. The deflection at the end of the beam is prevented by steel rods fastened from above. Find the size of rods necessary and the maximum fiber stress in the beams.

Solution.—$W = 200 \times 6 \times 10 = 12{,}000$ lb.
The tension in the rod

$$R = \tfrac{3}{8}W = \tfrac{3}{8} \times 12{,}000 = 4{,}500$$

Let d be the diameter of the rod.

$$\frac{\pi d^2}{4} \times 16{,}000 = 4{,}500$$

$$d^2 = 0.36$$

$$d = 0.6 \text{ in.}$$

The commercial size may now be chosen. The maximum M is $Wl/8$. Also, section modulus = 8.7.

$$\frac{12{,}000 \times 6 \times 12}{8} = p \times 8.7$$

$$p = 12{,}420 \text{ lb.}$$

Example 2.—A beam, 10 in. wide, is 14 in. deep with a 20-ft. span. It is fixed at one end and supported at the other and carries a load of 400 lb. per lin. ft. The support settles ½ in. under the action of the load. Find the reaction of the support and the maximum fiber stress.

Hint.—If the support were removed, the deflection of the end would be

$$d_1 = +\frac{1}{8}\frac{Wl^3}{EI}$$

The support brings the end up.

$$d_2 = \frac{1}{3}\frac{Rl^3}{EI}$$

Then

$$d_1 = d_2 + \tfrac{1}{2}\text{ in.}$$

Now assume that the support does not settle. Find maximum fiber stress, and compare with the first case.

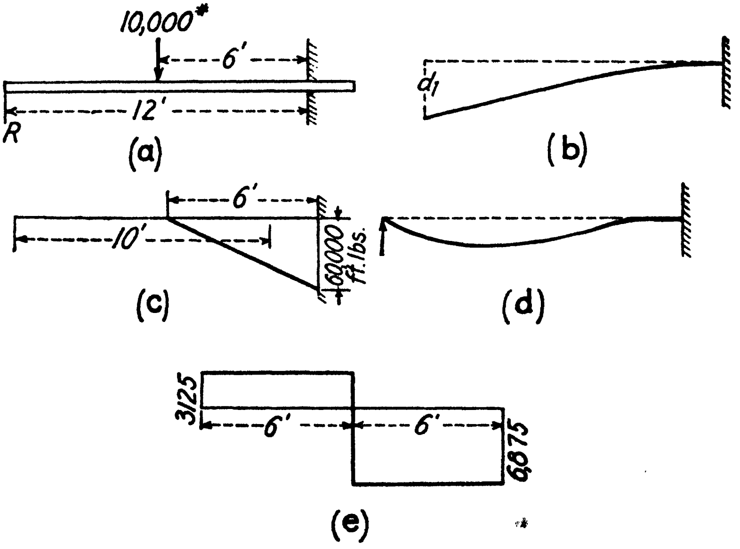


Fig. 405.—(*a*) Diagram of beam; (*b*) deflection due to load; (*c*) moment diagram; (*d*) elastic curve; (*e*) shear diagram.

Example 3.—A beam has one end fixed and the other end resting on a support. The span equals 12 ft. The beam carries a concentrated load of 10,000 lb. at the center. Select the smallest I beam to carry the load (Fig. 405).

Solution.—If the support were removed, the beam would deflect as shown in Fig. 405*b*. The deflection d_1 will now be calculated by

Eq. (55*a*). Figure 405*c* shows the moment diagram, which is a triangle. In calculating d_1, all dimensions must be in inches. Then

$$d_1 = \frac{120}{EI} \times \frac{1}{2} \times 72 \times 60{,}000 \times 12$$

Figure 405 shows the support replaced by a force R which brings the end of the beam back to the original level.
By Eq. (55),

$$d_1 = \frac{1}{3}\frac{R(144)^3}{EI}$$

Equating these two values of d_1 results in

$$R = 3{,}125 \text{ lb.}$$

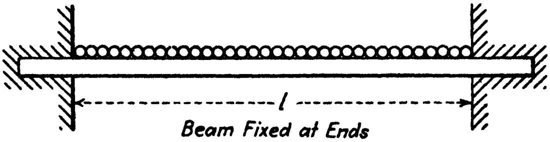
Figure 405*e* shows the shear diagram from which the maximum bending moment is

$$M = 6 \times 3{,}125 - 6 \times 6{,}875 = -22{,}500 \text{ ft.-lb.}$$

Substituting in the bending-moment equation,

$$22{,}500 \times 12 = 16{,}000 \times \text{section modulus}$$
$$\text{Section modulus} = 17$$

From any steel handbook, an 8-in. 25.25-lb. I beam will be found suitable.



Beam Fixed at Ends

FIG. 406.

177. Beam with Fixed Ends, Carrying a Uniformly Distributed Load.

—By means of calculus, the maximum bending moment for this beam is found to be

$$M = -\frac{Wl}{12} \tag{65}$$

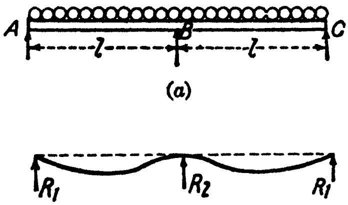
However, the assumption that the ends of the beam are rigidly fixed, is not always realized. More often, the designer uses $Wl/10$, or even $Wl/8$, which would be on the side of safety. In many cases, the specifications prescribe the values to use for M.

The deflection for the beam is

$$y_{\max} = -\frac{Wl^3}{384EI}$$

This shows that a beam with fixed ends, when the ends are perfectly fixed, is much stiffer, as well as stronger, than a simple beam of equal span.

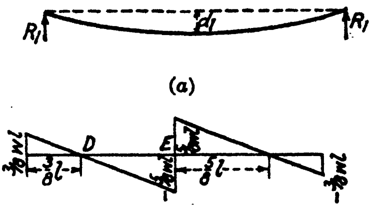
178. Beam Resting on Three Supports Equally Spaced. Figure 407*a* shows a beam with two equal spans of length *l*, carrying *w* lb. per lin. ft. The three supports are on the same level. If the middle support were removed, the beam would be as in Fig. 408*a* with a deflection



(*a*)

(*b*)

FIG. 407.



(*a*)

(*b*)

FIG. 408.

$$d = -\frac{5}{384}\frac{W(2l)^3}{EI}$$

where $W = 2lw$. If, now, a single force R_2 is applied at the center to bring that point back up to the original level the upward deflection is

$$d = \frac{1}{48}\frac{R_2(2l)^3}{EI}$$

Then

$$\frac{1}{48}\frac{R_2(2l)^3}{EI} = \frac{5}{384}\frac{W(2l)^3}{EI}$$

$$R_2 = \tfrac{5}{8}W = \tfrac{5}{4}wl$$

Since the end reactions are equal,

$$2R_1 = \tfrac{3}{4}wl$$

$$R_1 = \tfrac{3}{8}wl$$

With the reactions known, the shear diagram is drawn (Fig. 408*b*). The maximum bending moment is at E and is

$$M = \tfrac{9}{128}wl^2 - \tfrac{25}{128}wl^2 = -\tfrac{1}{8}wl^2$$

If the beam were cut through at E and made into two simple beams, the maximum B.M. for each part would be $+\frac{1}{8}wl^2$. Then the continuous beam is no stronger than two simple beams.

179. General Moment Equation.—Figure 409 shows a beam with supports R_1 and R_2. Let P_1 be the resultant of all loads to the left of R_1.

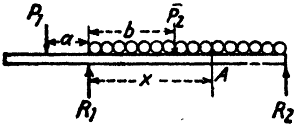


FIG. 409.

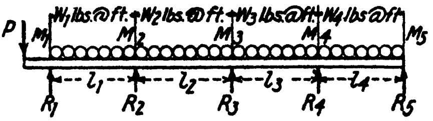


FIG. 410.

Taking moments around point A,

$$M_A = -P_1(a + x) + R_1x - w\frac{x^2}{2} - P_2(x - b)$$

$$M_A = -P_1a - P_1x + R_1x - w\frac{x^2}{2} - P_2(x - b)$$

$$= -P_1a + (R_1 - P_1)x - w\frac{x^2}{2} - P_2(x - b)$$

But P_1a equals the moment of all loads to the left of R_1, about R_1. Call this moment M_o.

Also, $R_1 - P_1 = V_1$, the shear at the right edge of the support. Then

$$M_A = M_O + V_1x - w\frac{x^2}{2} - P_2(x - b) \qquad (66)$$

This is called the *general moment equation.* By means of it, when the moment at a support is known, the shear at the right of that support may be determined. The reactions may now be found and the shear diagram constructed. the danger section of the beam being thus located.

180. Continuous Beams.—When a beam rests on three or more supports, it is called a *continuous beam.* Figure 410 shows a beam with five supports. In this volume, the discussion will be limited to a uniformly distributed load between supports. In more advanced works on strength of materials, the following equation known as the *theorem of three moments* is derived for the relation between the moments at three consecutive supports and the loads and spans between these supports, *viz.*,

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -\tfrac{1}{4}(w_1l_1^3 + w_2l_2^3) \qquad (67)$$

Similarly,

$$M_2l_2 + 2M_3(l_2 + l_3) + M_4l_3 = -\tfrac{1}{4}(w_2l_2^3 + w_3l_3^3)$$

and so on, for any three consecutive supports. By means of these relations, enough equations can be found to solve for the moments M_1, M_2, etc. Then, by means of the general moment equation (66), the shear at the right of each support may be found. The shear diagram may now be constructed, the danger sections located, and the maximum bending moment computed.

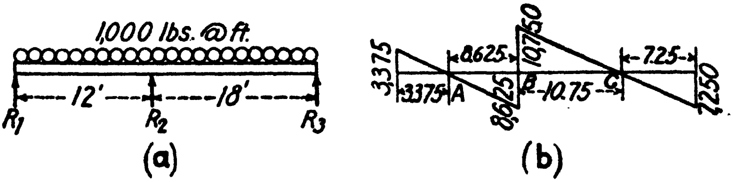


Fig. 411.

Example 1.—A beam rests on three supports. The spans are 12 ft. and 18 ft., respectively, and it carries 1,000 lb. per lin. ft. Design an I beam to carry the load (Fig. 411).

By inspection,

$$M_1 = M_3 = 0, \qquad \text{also } l_1 = 12 \text{ ft.}, \qquad l_2 = 18 \text{ ft.}$$

and

$$w_1 = w_2 = 1\,000 \text{ lb.}$$

Substituting in Eq. (67),

$$0 + 2M_2(12 + 18) = -\tfrac{1}{4} \times 1{,}000(12^3 + 18^3)$$
$$M_2 = -31{,}500 \text{ ft.-lb.}$$

Referring to the general moment equation (66),

$$M_A = M_2, \qquad M_0 = M_1, \qquad x = 12$$

and $P_2 = 0$. Making these substitutions,

$$-31{,}500 = 12V_1 - \frac{1{,}000}{2}(12)^2$$
$$V_1 = 3{,}375 \text{ lb.}$$

Notice that V_1 is the shear at the right edge of the left support. For the second span,

$$M_A = M_3 = 0, \qquad M_0 = M_2,$$

$x = 18$, and $V = V_2$, the shear at the right edge of the second support. Then

$$0 = -31{,}500 + 18V_2 - \frac{1{,}000}{2}(18)^2$$
$$V_2 = 10{,}750 \text{ lb.}$$

Next, construct the shear diagram, Fig. 411*b*.

$$M_A = \frac{3{,}375 \times 3.375}{2} = +5{,}695 \text{ ft.-lb.}$$
$$M_B = M_A - \frac{8{,}625 \times 8.625}{2} = -31{,}500 \text{ ft.-lb.}$$
$$M_C = +\frac{7{,}250 \times 7.25}{2} = +26{,}280 \text{ ft.-lb.}$$

Maximum $M = -31{,}500$ ft.-lb.

$$31{,}500 \times 12 = 16{,}000 \times \text{section modulus}$$
$$\therefore \text{Section modulus} = 23.63$$

Then a 9-in. 35-lb. or a 10-in. 25.4-lb. I beam will carry the load. Let V_1^1, V_2^1, and V_3^1 be the values of the shear at the left edge of the supports R_1, R_2, and R_3, respectively. Then

$$V_1^1 + R_1 = V_1$$

But

$$V_1^1 = 0$$

Then

$$R_1 = V_1 = 3{,}375 \text{ lb.}$$
$$V_2^1 + R_2 = V_2$$
$$R_2 = 10{,}750 - (-8{,}625) = 19{,}375 \text{ lb.}$$

Also,

$$R_3 + V_3^1 = V_3 = 0$$
$$R_3 = -(-7{,}250) = 7{,}250 \text{ lb.}$$

Check:

$$3{,}375 + 19{,}375 + 7{,}250 = 30{,}000 \text{ lb.}$$

Example 2.—A beam has three equal spans of 15 ft. each, and overhangs each support 6 ft. The left overhang carries 2,000 lb. at the end and the right overhang a uniformly distributed load of 400 lb. per ft. The distributed load on the spans are as follows: 500 lb. per ft. on the first, 700 lb. on the second, and 1,000 lb. on the third. Select an I beam to carry the load (Fig. 412).

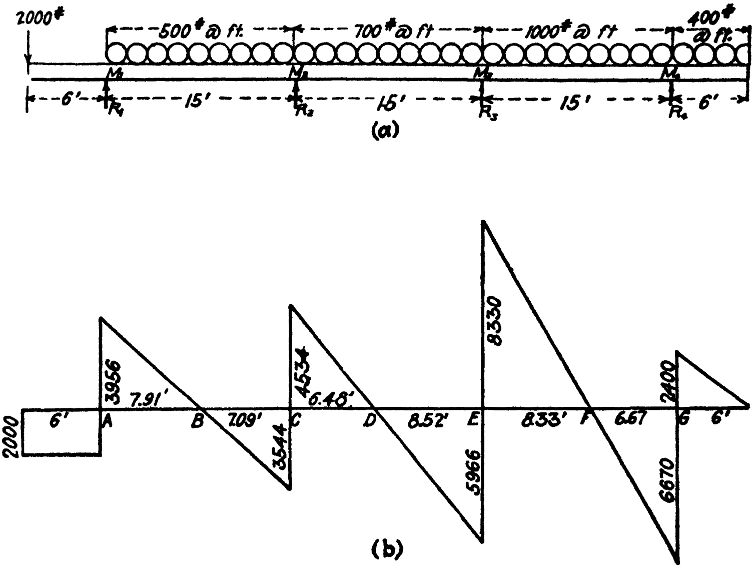


Fig. 412.

Solution.—By inspection,

$$M_1 = -6 \times 2{,}000 = -12{,}000 \text{ ft.-lb.}$$
$$M_4 = \frac{-400}{2}(6)^2 = -7{,}200 \text{ ft.-lb.}$$

Substituting in Eq. (67),

$$-12{,}000 \times 15 + 2M_2(15 + 15) + 15M_3 = -\tfrac{1}{4}(500 \times \overline{15}^3 + 700 \times \overline{15}^3)$$

$$15 \times M_2 + 2M_3(15 + 15) - 15 \times 7{,}200 = -\tfrac{1}{4}(700 \times \overline{15}^3 + 1{,}000 \times \overline{15}^3)$$

Simplifying these equations,

$$4M_2 + M_3 = -55{,}500 \text{ ft.-lb.}$$
$$M_2 + 4M_3 = -88{,}425 \text{ ft.-lb.}$$

Solving simultaneously, we have

$$M_2 = -8{,}910 \text{ ft.-lb.,} \qquad M_3 = -19{,}860 \text{ ft.-lb.}$$

From the general moment equation (66),

$$-8{,}910 = -12{,}000 + 15V_1 - \tfrac{500}{2}(15)^2$$
$$-19{,}860 = -8{,}910 + 15V_2 - \tfrac{700}{2}(15)^2$$
$$-7{,}200 = -19{,}860 + 15V_3 - \frac{1{,}000}{2}(15)^2$$
$$0 = -7{,}200 + 6V_4 - \tfrac{400}{2}(6)^2$$

Solving these four equations produces

$$V_1 = 3{,}956 \text{ lb.} \qquad V_2 = 4{,}534 \text{ lb.}$$
$$V_3 = 8{,}330 \text{ lb.} \qquad V_4 = 2{,}400 \text{ lb.}$$

The shear diagram may now be drawn as shown in Fig. 412*b*. From the relation,

$$V_1^1 + R_1 = V_1, \qquad V_2^1 + R_2 = V_2, \text{ etc.}$$
$$-2{,}000 + R_1 = 3{,}956, \qquad 3{,}544 + R_2 = 4{,}534$$
$$R_1 = 5{,}956 \text{ lb.,} \qquad R_2 = 8{,}078 \text{ lb.}$$
$$R_3 = 14{,}926 \text{ lb.,} \qquad R_4 = 9{,}070 \text{ lb.}$$

As a check on the work, the sum of the reactions equals the total load. The moments at the danger sections will now be calculated.

$$M_A = -12{,}000 \text{ ft.-lb.}$$
$$M_B = -12{,}000 + \frac{3{,}956 \times 7.91}{2} = 3{,}650 \text{ ft.-lb.}$$
$$M_C = 3{,}650 - \frac{3{,}544 \times 7.09}{2} = -8{,}910 \text{ ft.-lb.}$$
$$M_D = -8{,}910 + \frac{4{,}534 \times 6.48}{2} = 5{,}780 \text{ ft.-lb.}$$
$$M_E = 5{,}780 - \frac{5{,}966 \times 8.52}{2} = 19{,}644 \text{ ft.-lb.}$$
$$M_F = -19{,}644 + \frac{8{,}330 \times 8.33}{2} = 15{,}050 \text{ ft.-lb.}$$
$$M_G = 15{,}050 - \frac{6{,}670 \times 6.67}{2} = -7{,}200 \text{ ft.-lb.} = M_4$$

Then the maximum bending moment is −19,644 ft.-lb.

$$16{,}000 \times \text{section modulus} = 19{,}644 \times 12$$
$$\text{Section modulus} = 14.75.$$

Use an 8-in. 20.5-lb. I beam.

181. Three-moment Theorem.—In a very large number of actual cases the span lengths will be the same and the loads will be uniform throughout. Equation (67) will then simplify to the following:

$$M_1 l + 2M_2 \times 2l + M_3 l = -\tfrac{1}{4}(wl^3 + wl^3)$$

Canceling l, we have

$$M_1 + 4M_2 + M_3 = -\tfrac{1}{4}(wl^2 + wl^2)$$
$$M_1 + 4M_2 + M_3 = -\tfrac{1}{2}wl^2 \qquad (68)$$

This simplified equation may be written for successive supports as in the solution of Example 2, Art. 180.

182. Theorem of Three Moments for Concentrated Loads.—The following equation for concentrated loads on a continuous beam is derived in much the same manner as the equation for distributed loads. The derivation will be taken for granted. Figure 413 will make clear the various factors involved. Then

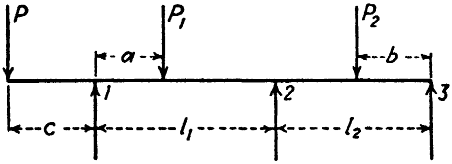


FIG. 413.

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\frac{P_1 a(l_1^2 - a^2)}{l_1} - \frac{P_2 b(l_2^2 - b^2)}{l_2}$$

Although this equation may be used less frequently than the previous one for distributed loads, yet the problems involving its use are equally important. Thus, the forces due to the thrust of the connecting rods against the crankshaft of an internal-combustion engine furnish a good illustration of a problem the solution of which requires the determinations of the bearing reactions as well as the bending moments due to concentrated loads. It is well known that these thrusts act intermittently at various points along the crankshaft and, therefore, in different spans. The distance between adjacent bearings is called a *span.*

In many of the problems, Fig. 413 will be simplified if the spans and the loads are the same throughout, with no overhang at the ends. Usually, $a = b = l_1/2$.

With this uniformity, the equation simplifies to the following:

$$M_1l_1 + 2M_2(2l_1) + M_3l_1 = -\frac{P_1(l_1/2)[l_1^2 - (l_1/2)^2]}{l_1} - \frac{P_2(l_1/2)[l_1^2 - (l_1/2)^2]}{l_1}$$

$$M_1l_1 + 2M_2(2l_1) + M_3l_1 = -\frac{3P_1l_1^2}{8} - \frac{3P_2l_1^2}{8}$$

Canceling l_1, we have

$$M_1 + 4M_2 + M_3 = -\frac{3P_1l_1}{8} - \frac{3P_2l_1}{8}$$

If $P_1 = P_2$,

$$M_1 + 4M_2 + M_3 = -2 \times \frac{3P_1l_1}{8} = -\frac{3P_1l_1}{4} \qquad (69)$$

183. Computation of Reactions.—In the foregoing examples in Art. 180 the reactions have been computed by first determining the shears on opposite sides of the supports. The reactions will then be equal to the sum of the respective shears without regard to algebraic sign. From the shear diagram, it can be seen that the reaction at any point may be thought of as the sum of two line segments as shown in Fig. 412.

In Fig. 411, assume the left span as the free body. R_1 may be determined by taking moments about R_2. Since there is a moment M at R_2, the difference between the clockwise and the counterclockwise moments will be equal to M. Thus,

$$12R_1 - 1{,}000 \times 12 \times 6 = -31{,}500$$

$$R_1 = \frac{-31{,}500 + 72{,}000}{12} = 3{,}375$$

In Fig. 412, take the portion of the beam to the left of R_2 as the free body, and take moments about R_2. As before,

$$15R_1 - 2{,}000 \times 21 - 500 \times 15 \times 7.5 = M = -8{,}910$$

$$R_1 = \frac{-8{,}910 + 42{,}000 + 56{,}250}{15} = 5{,}956 \text{ lb.}$$

The operation may be repeated for determining R_2 of Fig. 412 by taking the beam to the left of R_3 as the free body. Thus,

$$5{,}956 \times 30 + R_2 \times 15 - 2{,}000 \times 36 - 500 \times 15 \times 22.5 - 700 \times 15 \times 7.5 = -19{,}860$$

$$R_2 = \frac{-19{,}860 - 178{,}680 + 72{,}000 + 168{,}750 + 78{,}750}{15} = 8{,}074 \text{ lb.}$$

Additional reactions may be computed as required by repeating the method illustrated above.

Example.—In Fig. 413, let $l_1 = l_2 = 5$ ft. There is no overhang at either end. Let $P_1 = P_2 = 800$ lb. What will be the moments at the supports? What are the reactions?

Solution.—$M_1 = 0$, $M_3 = 0$, for this case.

From Eq. (69),

$$0 + 4M_2 + 0 = -\tfrac{3}{4} \times 800 \times 5$$

$$M_2 = -\tfrac{3}{16} \times 4{,}000 = -750 \text{ lb./ft. } \textit{Ans.}$$

To find the reaction R_1, take moments about R_2. Then

$$R_1 \times 5 - 800 \times 2.5 = -750$$

$$R_1 = \frac{-750 + 2{,}000}{5} = 250 \text{ lb.}$$

From symmetry,

$$R_1 = R_3 = 250 \text{ lb.} \qquad R_2 = 2 \times 800 - 2 \times 250 = 1{,}100 \text{ lb. } \textit{Ans.}$$

Problems

344. Two 10-in. 25-lb. channels and two 10- by ½-in. cover plates are used to form a plate girder 12 ft. long. What total uniformly distributed load on this simple span will produce a deflection not in excess of 0.4 in.? What will be the bending stress due to this load?

345. In Prob. 344, what would be the bending stress due to the load if an additional support were placed at the middle, with all supports at the same level? Sketch the shear and moment diagrams.

346. A beam 28 ft. long is supported at both ends and at a point 12 ft. from the left end. The beam carries a load of 600 lb. per lin. ft. Sketch the shear and moment diagrams. What size of I beam should be used?

347. What size of I beam should be used for a beam 36 ft. long and supported so that there are four equal spans? The beam carries a load of 640 lb. per lin. ft. What are the amounts of the moments at each support, and the reactions at the supports?

348. An I beam 20 ft. long is supported so that there are two spans of 10 ft. each. It carries a load including its own weight of 600 lb. per lin. ft. What is the proper size of I beam? Sketch the shear and moment diagrams.

349. If the beam in Prob. 348 were cut so as to make two simple spans, what load would it be able to carry?

350. What are the bending and shearing stresses in a beam 9 ft. long, fixed at one end and supported at the same level at the other? The cross section of the beam is 6 by 12 in. It carries a load including its own weight of 820 lb. per lin. ft. Sketch the shear and moment diagrams. $E = 1{,}200{,}000$.

351. If the end support in Prob. 350 settles 0.4 in., compute the reactions and bending and shearing stresses.

352. If the end support in Prob. 350 is pushed 0.4 in. above its previous level, what will be the stresses in the beam?

353. A 12-in. 35-lb. I beam is 20 ft. long. What is the maximum bending stress due to a load of 840 lb. per lin. ft., including its own weight? The beam is supported at the ends and at the middle.

354. What would be the bending stress in the beam of Prob. 353 if the end supports settled $\frac{1}{2}$ in.?

355. Compare the moments, respectively, in: a simple beam; a beam supported at the ends and at the middle, a beam fixed at one end and freely supported at the other.

184. Reinforced Concrete.—In beams of the same material throughout, the equation $p = Mc/I$ is used to determine the stress or to find the size of the beam. In this equation, it is assumed that the modulus of elasticity in compression is the same as in tension. Furthermore, the neutral axis of the section always coincides with the gravity axis of the section.

In the reinforced-concrete beam, two materials having different moduli of elasticity are used to form a unit. The concrete has a high resistance to compression but can withstand very little tension. Steel is therefore placed on the tension side of the beam and is expected to resist all the tension. The concrete on the tension side of the neutral axis merely serves to hold the steel in place. It must be evident that the neutral axis does not coincide with the gravity axis of the cross section.

In consequence of the foregoing conditions, a new equation must be derived to take the place of $p = Mc/I$. The following assumptions are made: The unit deformation is proportional to the distance from the neutral surface. The stresses are within the elastic limit. The steel provides all the tensile resistance.

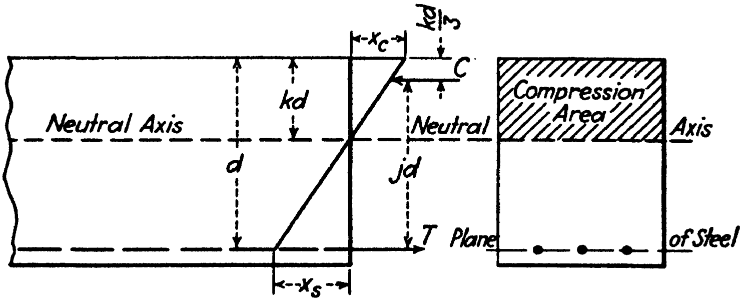


FIG. 414.

In Fig. 414, it can be seen that the steel is placed a short distance above the bottom of the beam. Such placing serves a double purpose. The steel is protected against fire, and, secondly, the concrete grips the steel, holding it firmly in position, so that its tensile resistance may be developed. For purposes of computation, the effective depth of the beam is taken as the distance from the center of gravity of the steel to the extreme compressive fiber of the concrete.

Since the total compression is distributed over the cross section of the concrete above the neutral axis and the total tension is carried by the cross-sectional area in the plane of the steel, it is evident that the position of the neutral axis must be found before the moment arm of the stress couple can be determined. In Fig. 414, let x_c be the unit deformation in the outer fiber of the concrete, x_s the unit deformation in the steel, S_c the stress in the concrete, and S_s the stress in the steel. Then by proportion,

$$\frac{x_c}{x_s} = \frac{kd}{d - kd}$$

But

$$x_c = \frac{S_c}{E_c} \qquad \text{and } x_s = \frac{S_s}{E_s}$$

$$\therefore \frac{S_c/E_c}{S_s/E_s} = \frac{kd}{d - kd}$$

Then

$$\frac{S_c}{S_s} \times \frac{E_s}{E_c} = \frac{kd}{(1 - k)d}$$

Let

$$\frac{E_s}{E_c} = n$$

Then

$$\frac{nS_c}{S_s} = \frac{k}{1 - k}$$

On any section the total tension must equal the total compression. The total compression may be written as

$$C = \frac{S_c}{2}bkd$$

and the total tension as

$$T = A_sS_s$$

The value of A_s, the area of the steel, is expressed as a percentage of the area of the cross section bd of the beam or

$$A_s = pbd$$

Total tension may then be written

$$T = pbdS_s$$

Since $C = T$, it follows that $(S_c/2)bkd = pbdS_s$. The bd being canceled on each side of the equation, this reduces to

$$\frac{S_c}{2}k = pS_s$$

$$S_c = \frac{2pS_s}{k}$$

Substituting this value of S_c in the equation $k/(1 - k) = nS_c/S_s$ changes this to

$$\frac{k}{1 - k} = \frac{2pnS_s}{kS_s}$$

If the S_s is canceled and fractions are cleared, this equation assumes the form

$$k^2 = 2pn - 2pnk$$

which may now be solved for k after first transposing all terms to the left side and equating to zero. From the binomial theorem, any quadratic may be solved by means of the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Since k represents x,

$$k = \frac{-2pn \pm \sqrt{4p^2n^2 + 8pn}}{2}$$

Now, taking out the square root of 4 from under the radical produces

$$k = \sqrt{p^2n^2 + 2pn} - pn \tag{70}$$

The value of k can now be determined if p and n are known or assumed. Since kd is the distance from the extreme fiber in compression to the neutral surface, the position of the neutral axis is known when the value of k is known.

An average value of k is ⅜, and this value will be used in the examples that are to follow.

From Fig. 414, it can be seen that the total compressive force may be assumed as concentrated at a distance of $kd/3$ from the extreme fiber in compression. The distance between C and T is equal to

$$d - \frac{kd}{3} = d\left(1 - \frac{k}{3}\right)$$

The term $\left(1 - \frac{k}{3}\right)$ is called j, and the moment arm is then designated as jd.

The average value of j is $\frac{7}{8}$, and this will be used in the problems.

As in the case of steel or wooden beams, the resisting moment is equal to the moment of the stress couple. That is,

$$M = Tjd \qquad \text{or } M = Cjd$$

The values of T and C being substituted, the equations become

$$M = pjS_sbd^2 \tag{71}$$

$$M = \frac{S_c}{2}kjbd^2 \tag{72}$$

Either of these expressions for M represents the resisting moment of the beam..

Example 1.—A simple beam 10 ft. long carries a load of 1,000 lb. per lin. ft., including its own weight. If $p = 0.8$ per cent and the stresses in the steel and concrete shall not exceed 16,000 and 650 lb. per sq. in., respectively, what should be the size of the beam? Let $b = \frac{3}{4}d$.

Solution.

$$M = \frac{Wl \times 12}{8} = \frac{1{,}000 \times 10 \times 10 \times 12}{8} = 150{,}000 \text{ in.-lb.}$$

Then

$$pjS_sbd^2 = 150{,}000$$

Using the designated values and solving,

$$bd^2 = \frac{150{,}000}{0.008 \times \frac{7}{8} \times 16{,}000} = 1{,}340$$

and

$$\begin{aligned} \tfrac{2}{3}d^3 &= 1{,}340 \\ d &= 12.65 \text{ in.} \end{aligned}$$

This is the effective depth, or the depth to the center of the steel. To provide for the covering of the steel, a total depth of 14 in. may be assumed.

The resisting moment of the concrete will now be used in designing the beam. Then

$$\frac{S_c}{2}kjbd^2 = 150{,}000$$

and, allowable values being used,

$$\begin{aligned} bd^2 &= \frac{2 \times 150{,}000}{650 \times \frac{3}{8} \times \frac{7}{8}} = 1{,}407 \\ \tfrac{2}{3}d^3 &= 1{,}407 \\ d^3 &= 2{,}109 \\ d &= 12.8 \text{ in.} \end{aligned}$$

This depth is greater than that obtained when the resisting moment of the steel is used in the computation. The larger value will be used, and the beam will be

8½ in. by 14 in.

185. Ideal Steel Ratio.—In the example of Art. 184, the percentage of steel was assumed. When this assumption is made the allowable values of stress in steel and concrete will not occur at the same time. That is, when the resisting moment of the steel is used in designing the beam, then the concrete stress may be greater or less than the allowable value indicated. To illustrate, assume 1 per cent of steel. Since $C = T$,

$$\frac{S_c}{2}kbd = pbdS_s$$

Canceling the bd on both sides,

$$\frac{S_c}{2}k = pS_s$$

Substituting values for k and p and solving,

$$\frac{S_c}{2} = \frac{0.01S_s}{0.375} = \frac{10}{375}S_s$$

The average unit stress in the concrete is $\frac{10}{375}$ as great as the stress in the steel. On this basis, if the steel were stressed to 16,000 lb. per sq. in., the corresponding average stress in the concrete would be

$$\tfrac{10}{375} \times 16{,}000 = 426 \text{ lb.}$$

and the extreme fiber stress in the concrete would be 2 × 426 = 852 lb. per sq. in., a value that is much in excess of what was allowed in the example.

If it is desired to design a beam in which both the steel and the concrete are to be stressed to certain definite values, a steel ratio must be determined for the allowable stresses. Since $T = C$,

$$S_s pbd = \frac{S_c}{2} kbd$$

Solving, we have

$$p = \frac{S_c k}{2S_s}$$

Let $k = \frac{3}{8}$; then

$$p = \frac{3S_c}{16S_s}$$

Let the allowable stresses in the steel and the concrete be 16,000 and 650 lb. per sq. in., respectively; then the corresponding percentage of steel

$$p = \frac{3 \times 650}{16 \times 16{,}000} \times 100 = 0.76 \text{ per cent}$$

This is the ideal percentage of steel, and, when this amount is used in a beam loaded to its capacity, the maximum stresses will always correspond to the values assumed. If this percentage of steel is exceeded, the beam is over-reinforced, the concrete will reach its allowable stress first, and the strength of the beam depends on the concrete. If the percentage of steel is less than the ideal amount, then the beam is under-reinforced, and the strength of the beam depends on the steel.

Example 1.—A simple beam 12 ft. long supports a uniform load including its own weight of 1,000 lb. per ft. When the stresses above are used, what should be the dimensions of the beam if $b = \frac{2}{3}d$?

Solution.—The ideal steel ratio is worked out as before. Either moment equation may be used to determine bd^2.

$$M = \frac{Wl}{8} = \frac{1{,}000 \times 12 \times 12 \times 12}{8} = 216{,}000 \text{ in.-lb.}$$

$$\frac{S_c}{2}kjbd^2 = 216{,}000$$

$$bd^2 = \frac{216{,}000 \times 2}{650 \times \frac{3}{8} \times \frac{7}{8}} = 2{,}020$$

$$\frac{2d^3}{3} = 2{,}020$$

$$d^3 = \frac{2{,}020 \times 3}{2} = 3{,}030$$

$$d = 14.5 \text{ in.}$$

$$b = \tfrac{2}{3} \times 14.5 = 9.5 \text{ in.}$$

If 1½ in. of concrete below the center of the steel is allowed, the total depth would be 16 in. and the beam would be 9½ by 16 in. The area of steel is next determined.

$$A_s = pbd, \qquad A_s = 0.0076 \times 9.5 \times 14.5 = 1.04 \text{ sq. in.}$$

Four bars ½ in. square will give the area needed.

Example 2.—A simple beam 10 ft. long has an effective cross section of 9 by 12 in., with two ¾-in. round reinforcing bars. The allowable unit stresses in the steel and the concrete are 16,000 and 650 lb. per sq. in., respectively. What total uniformly distributed load will the beam support?

Solution.—Determine the ideal steel ratio.

$$\text{Ideal } p = \frac{3}{16} \times \frac{650}{16{,}000} = 0.0076$$

$$\text{Actual } p = \frac{2 \times 0.44}{9 \times 12} = 0.00815$$

By comparing the values of p, it can be seen that the beam is over-reinforced, and its strength depends on the concrete.

$$M = \frac{Wl}{8} \times 12 = \frac{S_c}{2}kjbd^2$$

Substituting all known values and solving,

$$W = \frac{650}{2} \times \frac{3}{8} \times \frac{7}{8} \times \frac{9 \times 144 \times 8}{12 \times 10} = 9{,}230 \text{ lb.}$$

Example 3.—A rectangular reinforced-concrete beam is 8 in. wide, and 12 in. deep from the compressive fiber to the center of the steel. The length is 20 ft. There are four ½-in. square bars in the same plane. What will be the unit stresses if the beam carries a load of 240 lb. per ft., including the weight of the beam?

$$M = \frac{Wl}{8} \times 12 = \frac{240 \times 20 \times 20 \times 12}{8} = 144{,}000 \text{ in.-lb.}$$

Then

$$\frac{S_c}{2} \times \frac{3}{8} \times \frac{7}{8} \times 8 \times 12^2 = 144{,}000$$

and

$$S_c = \frac{144{,}000 \times 2 \times 8 \times 8}{3 \times 7 \times 8 \times 144} = 762 \text{ lb./sq. in.}$$

For the steel,

$$S_s \times 1 \times \tfrac{7}{8} \times 12 = 144{,}000$$

Then

$$S_s = \frac{144{,}000 \times 8}{7 \times 12} = 13{,}700 \text{ lb./sq. in.}$$

Problems

356. What should be the size of a concrete beam 16 ft. long to support a uniform load of 450 lb. per ft., including its own weight? The allowable stresses are 16,000 and 750 lb. per sq. in., respectively. Let $b = \frac{2}{3}d$.

357. Compute the thickness of a concrete floor slab with a span of 8 ft. to support a live load of 150 lb. per sq. ft. Allowable stresses as in Prob. 356. The slab is continuous, and $M = Wl/12$.

358. A reinforced-concrete beam has a span of 18 ft. and is to support a live load of 900 lb. per lin. ft. The effective cross section is 8 by 18 in. The reinforcing consists of three ⅞-in. round bars in the same horizontal plane. The total depth of the beam is 20 in. What are the unit bending stresses?

186. Moment-of-inertia Method.

—A common method of dealing with composite structural members consists in obtaining a transformed section, in which one of the materials is replaced in terms of the other.

As shown in Art. 140, for equal deformations, the stresses in each of the two materials are proportional to their moduli of elasticity. From Fig. 414, it is seen that the total stress in the steel is $A_s s_s$. At this same distance from the neutral axis the total stress in the concrete would be $A_c s_c$. Since the force in the steel is to be replaced by an equal force in terms of concrete, then

$$A_s s_s = A_c s_c, \qquad \text{or } A_c = \frac{A_s s_s}{s_c} = nA_s$$

That is, nA_s is the area of the concrete that must take the place of and act in the plane of the steel in order to change the combined section into an equivalent homogeneous section. Figure 414*a* shows the transformed or equivalent section. Since the concrete below the neutral axis is not counted on to resist bending moment, the neutral axis of the transformed section will coincide with its gravity axis.

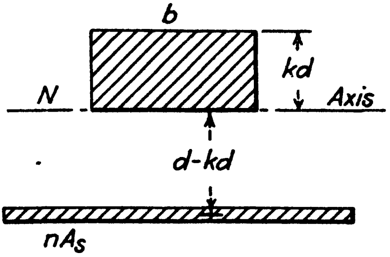


FIG. 414*a*.

Taking moments about the neutral axis,

$$nA_s(d - kd) = bkd\frac{kd}{2}$$

Since $A_s = pbd$,

$$npbd(d - kd) = bkd\frac{kd}{2}$$

$$npbd^2(1 - k) = bd^2\frac{k^2}{2}$$

Canceling bd^2,

$$2np - 2npk = k^2$$

Solving for k,

$$k = \sqrt{2pn + (pn)^2} - pn$$

Since k is known, the position of the neutral axis for given values of depth is known, and the moment of inertia can be determined.

Example 1.—Solve Example 2, of Art. 185, by the moment-of-inertia method. Also, find the deflection.

Solution.—$p = 0.00815$, $n = 15$.

$$k = \sqrt{2 \times 0.00815 + (0.00815 \times 15)^2} - 0.00815 \times 15 = 0.3877$$

The neutral axis will be at a distance $kd = 0.3877 \times 12 = 4.65$ in. The moment of inertia of the equivalent section will be

$$I = \frac{9 \times 4.65^3}{3} + 15 \times 2 \times 0.44 \times 7.35^2 = 1{,}013.7 \text{ in.}^4$$

The problem may now be solved by using the regular flexure formula, or

$$\frac{W \times 10 \times 12}{8} = \frac{650 \times 1{,}013.7}{4.65}$$

Solving,

$$W = 9{,}440 \text{ lb.}$$

This value does not agree with the answer to Example 2, in which k was assumed as $\frac{3}{8}$, with the result that an approximate but conservative answer was obtained.

Since all the deflection equations include the moment of inertia, the transformed section will always have to be used if the deflection of a reinforced-concrete beam is desired.

Example 2.—Compute the deflection of the reinforced-concrete beam in Example 1 above. Is the value within the allowable limits?

Solution.—The value of $I = 1{,}013.7$ in.^{4}, as already computed. For the deflection of a simple beam supporting a uniform load,

$$y = \frac{5Wl^3}{384EI}, \quad \text{or } y = \frac{5 \times 9{,}440 \times 120^3}{384 \times 2{,}000{,}000 \times 1{,}031.7} = 0.1048 \text{ in. } \textit{Ans.}$$

This value is seen to be well within the allowable limits of 0.33 in.

187. T Beams.—In the design of a common floor system the floor slab is supported by beams, and the beams are supported by girders. If the concrete for the slabs, beams, and girders is poured at one time, so that these parts form a unit, then it can be seen from Fig. 415 that a part of the slab may be considered as a part of the compression side of the beam and forms what is known as a "T beam." It is usual to consider an arbitrary length of the slab as a flange of the T beam. The projection of the flange on each side of the stem of the beam may be taken as equal to either six times the thickness of the slab or one-fourth of the span of the beam, the smaller value being taken.

In designing a T beam, it is usually found that there is a large excess of compression area. Consequently, the web of the beam is made large enough to resist the shear, and then sufficient tensile reinforcement is provided to resist the bending.

The necessary shear relations needed in determining the size of the web are derived as follows: Take a portion of a beam as a free body as shown in Fig. 416. The forces acting are the vertical shears V, the compressions C and C_1, and the tensions T and T_1. It is evident that, if the bending moment varies from section to section, then T_1 is not equal to T. $T - T_1$ is a resultant force which will produce a horizontal shear on any plane above the steel. Taking moments about the point P, we have

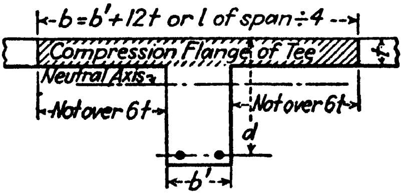


FIG. 415.

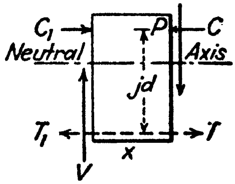


FIG. 416.

$$Vx = (T - T_1)jd, \qquad \text{or } \frac{V}{jd} = \frac{T - T_1}{x}$$

But $(T - T_1)/x$ is equal to the horizontal shear per linear inch of beam considered, and the unit shear is $(T - T_1)/x$ divided by b.

Then the unit shear

$$s_h = \frac{T - T_1}{bx} = \frac{V}{bjd}$$

From this equation it can be seen that

$$s_h = \frac{8}{7}\frac{V}{bd} \tag{73}$$

The usual value of s_h used is 120 lb. per sq. in. This value is in excess of what concrete can withstand, but the discrepancy is provided for by means of stirrups or web steel.

Example.—A simple beam 12 ft. long carries a load of 900 lb. per ft. including its own weight. The slab is 4 in. thick, and its span is 6 ft. The allowable shearing stress is 120 lb. per sq. in., and the steel and concrete stresses are 16,000 and 650 lb. per sq. in., respectively. It is required to determine the cross section of the stem of the T beam and the area of the tensile steel.

Solution.—Since

$$s_h = \frac{V}{bjd}$$

then

$$bd = \frac{V}{s_h j}$$

and

$$bd = \frac{900 \times 12}{2 \times 120 \times \frac{7}{8}} = 52$$

b may be assumed as equal to $\frac{1}{3}d$ to $\frac{1}{2}d$.
Let

$$b = \tfrac{1}{2}d$$

Then

$$\tfrac{1}{2}d^2 = 52$$
$$d^2 = 104$$

and

$$d = 10.2 \text{ in.}$$

which includes the thickness of the slab.
If 1¾ in. of concrete under the steel is allowed, a total depth of 12 in. may be used, and with even inches the width $b = 6$ in.

$$M = \frac{Wl}{8} = \frac{12 \times 900 \times 12 \times 12}{8} = 194{,}000 \text{ in.-lb.}$$

$$M = A_sS_sjd, \qquad A_s = \frac{194{,}400}{16{,}000 \times \frac{7}{8} \times 10.2} = 1.36 \text{ sq. in.}$$

Two bars, one $\frac{7}{8}$ in. round and one 1 in. round, will provide the required area of steel.

A check of the stress in the concrete will now be made. The effective width of the flange will be either,

$$b = (12 \times 4) + 6 = 54 \text{ in.,} \qquad \text{or } b = \tfrac{12}{4} \times 12 = 36 \text{ in.}$$

The moment arm of the concrete is taken as the distance from the center of the steel to the half depth of the slab, or

$$d - \tfrac{1}{2}t$$

$$\frac{S_c}{2}bt\left(d - \frac{1}{2}t\right) = 194{,}400$$

$$S_c = \frac{194{,}400 \times 2}{(36 \times 4)(10.2 - 2)} = 330 \text{ lb./sq. in.}$$

188. Shear and Diagonal Tension.—In proportioning the stem of the T beam, the expression $s_h = V/bjd$ was used.

At any point in the beam, there is a unit vertical shear and a unit horizontal shear of the same intensity, which produce tensile stresses on a diagonal plane and are known as "diagonal tension." Figure 416 may be changed to Fig. 417 if $T - T_1$ is replaced by an equivalent total shear. Then, on any section such as PQ below the neutral axis the shears will have components that produce tension on such sections. The tension due to the shear is most serious in the regions of large vertical shear. The actual amount of the diagonal tension is impossible to determine, for a slight tension due to bending should also be included. But the tensile stress in the concrete is not known, since it is assumed that the horizontal steel carries all the stress due to bending. The shear is therefore taken as a measure of the diagonal ten-

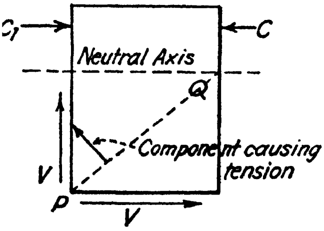


FIG. 417.

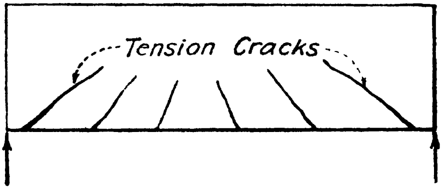


FIG. 418.

sion, and the unit shear is restricted to low values. Figure 418 shows the trend of tension cracks, and Fig. 419 shows how steel is placed to provide for the tension. Usually, some of the horizontal steel is bent up, but this practice is generally limited, so that additional U-shaped bars, called "stirrups," are provided to resist shear not cared for by the bent-up steel.

Theoretically, the steel bars, or stirrups, should cross the tension cracks at right angles, but this arrangement is not feasible, for the cracks are not parallel to each other and there could be no regular system of placing the bars. Further, it is difficult to hold inclined stirrups in place. For these reasons, they are generally placed in a vertical position.

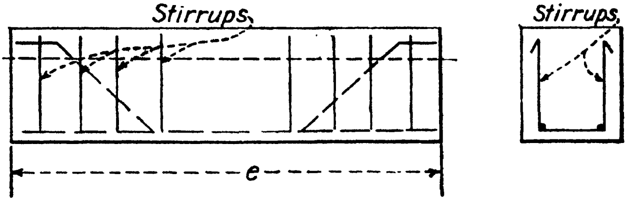


FIG. 419.

Some designers space the stirrups uniformly along the beam, placing them at distances equal to $d/2$ apart. However, it is better practice to space in accordance with the variation in the shear along the beam.

It was previously shown that V/jd is the horizontal shear per linear inch. For a length of beam P, the amount of horizontal shear is equal to VP/jd where V is the average total vertical shear for the length P. Since the horizontal and vertical shears are equal, VP/jd is equal to the vertical component of the diagonal tension. The horizontal component of the diagonal tension is assumed to be resisted by the horizontal steel, but the vertical component is resisted by the concrete and by vertical stirrups.

If A_s is the area of a stirrup, then

$$A_sS_s = \frac{V'P}{jd}$$

where V' is the portion of the total shear allotted to the steel. Solving this equation,

$$P = \frac{A_s S_s jd}{V'}$$

which represents the expression for computing the spacing of stirrups. There are various recommendations regarding the amount of shear that shall be taken by the stirrups. A common method consists in assuming that the concrete will carry a stress of 40 lb. per sq. in. and supplying steel to take care of the excess shear not carried by the concrete.

Example.—Investigate the shear and compute the spacing of stirrups needed for the simple beam in the example in Art. 187.

Solution.

$$V = \frac{900 \times 12}{2} = 5{,}400 \text{ lb.}$$

$$s_h = \frac{5{,}400}{6 \times \frac{7}{8} \times 10.2} = 103 \text{ lb./sq. in.}$$

If we assume that concrete carries 40 lb. per sq. in., then steel must be provided to carry 63 lb. per sq. in.

$$V' = \tfrac{63}{103} \times 5{,}400 = 3{,}300 \text{ lb.}$$

The diameter of the stirrup will be assumed so that the spacing will not be less than 4 in.; if the spacing should exceed $\frac{3}{4}d$, it would be arbitrarily limited to some value equal to or less than $\frac{3}{4}d$. Assume $\frac{5}{16}$-in. stirrup, for which

$$A_s = 2 \times 0.077 = 0.154 \text{ sq. in.}$$

$$P = \frac{0.154 \times 16{,}000 \times \frac{7}{8} \times 10.2}{3{,}300} = 6.66 \text{ in.}$$

A spacing of 6½ in. would be used for a distance of about 2 ft. The shear at a section 2 ft. from the support will be

$$\tfrac{4}{6} \times 5{,}400 = 3{,}600 \text{ lb.}$$

Of this amount, 2,100 lb. is taken by the concrete, 1,500 lb. being left for the stirrups.

$$P = \frac{0.154 \times 16{,}000 \times \frac{7}{8} \times 10.2}{1{,}500} = 14.7 \text{ in.}$$

This spacing is greater than the depth of the beam and would not be used, but a revised spacing equal to $\frac{3}{4}d$ or $7\frac{1}{2}$ in. would be used until the section where stirrups are no longer needed is reached.

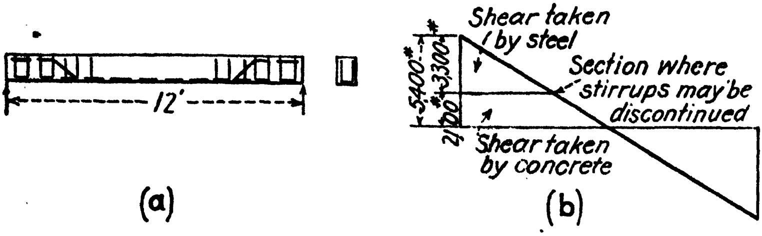


Fig. 420.

Figure 420 shows some of the details of the beam and also the shear diagram which may be used as an aid in designing the stirrups.

CHAPTER XVII

METAL TIMBER AND CONCRETE COLUMNS

189. Definitions.—A compression member of a structure or machine with its length at least ten times the least dimension in cross section is spoken of as a *column*. Under this classification will come the columns in a building, the compression members in a roof truss, and the piston and connecting rods in an engine, as well as thrust rods or bars in machines and the webs of I beams.

Slenderness Ratio.—The ratio of the length of the column to the radius of gyration of the cross section about an axis perpendicular to the plane of bending is called the *slenderness ratio.* It is indicated as (l/r) or (l/d). The value of r is the least radius of gyration. For columns rectangular in cross section, the ratio is frequently written as l/d, where d is the least transverse dimension.

For columns with a slenderness ratio of about 50, the failure of the column under load is due to direct compression; if the ratio is over 150, the failure is largely due to bending. A large number of formulas has been proposed for columns of intermediate lengths. However, such items as imperfections in structure, nonuniform material, initial bending, and uncertainty regarding end supports are important and do not appear as factors in any of the formulas.

Engineers do not agree as to what ratio of length to least dimension shall constitute a column. The ratios used vary all the way from 10 to 15. Experiment has shown that if the ratio mentioned is less than 10 and the load is applied centrally at the ends, if the member fails it is usually by direct crushing. Formulas for safe designs against crushing have been given in previous paragraphs. If the ratio of length to least dimension of cross section is 10 or more and the piece of material is centrally loaded until failure occurs, experiment has shown that the failure is due to a combination of crushing and bending or buckling sideways. Such a member is called a *long column* or simply a *column*. However, to be on the safe side, the student should not adhere to any hard fast rule for the ratio but should figure for crushing and buckling.

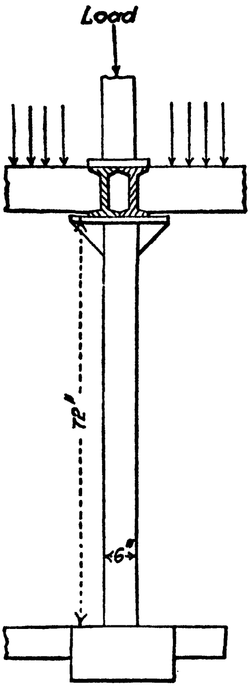
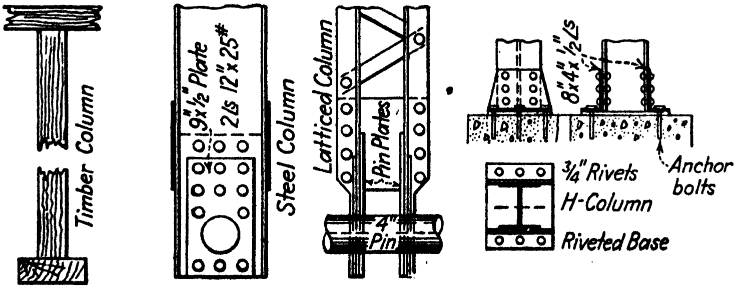


FIG. 421.

Figure 421 shows such a column under a load. The least dimension shown is 6 in., and the length is 6 ft., or 72 in. Since 72 in. is twelve times 6 in., the column is classified as a long column and must be figured as such.

Figure 422 shows the types of long column. A pin-connected column is not so strong as the other types, because of the ease with which the ends may turn about the pins. It is used more, experimentally, in determining the theory of columns.



Flat End Pin End Flat End

FIG. 422.

Columns with both ends fixed or both ends flat are the ones used in common practice, because of the ease of fastening or fixing them and also because they are stronger than the other types. A flat-end column must have the ends squared up so that the timber is at right angles to the surface against which it bears. This requirement means that the column would bear equally over all parts of its ends against the resisting surface and also that the load would bear equally over the top of the column and its resultant would act straight down through the center of the column.

The ends of a flat column must either be braced to prevent tipping or side motion or be riveted to the structure if this is made of steel. In practice, flat or fixed ends are figured as equally strong.

There are many column formulas in use, any one of which gives reliable results when the factor of safety is chosen that governs the special conditions under which the column is to be used. However, only a few of such formulas will be considered in this work. It is assumed in every case that the ends are squared and fixed in place and that the load is centrally applied. That is, it is assumed that the center of bearing area is the center of the column.

In selecting a column formula for figuring the columns of a building, the engineer is often restricted in his choice. If the building is to be constructed in cities, the building ordinances specify the formulas to be used. For many years, engineers used Euler's, Gordon's, and Rankine's formulas. These formulas are based more directly on theory that required a rather close observance of the "slenderness ratios l/r." Also, they are based on certain definite end conditions, fixed positions of loads, and a uniformity of materials. But such conditions are not realized in practice. Then, too, the above-mentioned formulas require tedious computations. Engineering practice makes extensive use of straight-line formulas. Experiments conducted for several years with all kinds of materials under average working conditions have shown that the straight-line formulas give results quite reliable.

The student will find that, although the formulas of various localities may differ, the difference will be in certain constants used. No trouble should be found in their application.

The type of column considered will carry a load assumed to be applied at the center. This load may come from another column, or from beams resting on the column, or from both.

For wooden columns, the Chicago Ordinance for years has specified the formula (called "Winslow's formula")

$$p = C\left(1 - \frac{L}{80D}\right) \tag{74}$$

Where p is the stress per square inch developed by the load; C is the working compressive stress of the lumber parallel to the grain, for short blocks; L equals the length in inches; and D the least diameter or dimension in inches.

The specifications further require that the maximum length shall not exceed 30 diameters—that is, L divided by D must not be greater than 30.

Another formula that has been extensively used in designing wooden columns is

$$\frac{P}{A} = C - 20\frac{L}{d} \tag{75}$$

where C is the allowable compressive stress for a short block of the timber to be used, and d is the least dimension of the cross section. This formula reduces to

$$\frac{P}{A} = 1{,}000 - 5.8\frac{1}{r}$$

for southern pine. d is replaced in terms of r.

Example 1.—Find a safe load that a 10-ft. column 6 by 6 ft. will carry (Fig. 423*a*).

Solution.—$D = 6$, and $A = 6 \times 6 = 36$ sq. in. A safe working compressive fiber stress parallel to the grain is 1,000 lb. Then $C = 1{,}000$, and $L = 10 \times 12 = 120$ in. L divided by D is $\frac{120}{6} = 20$.

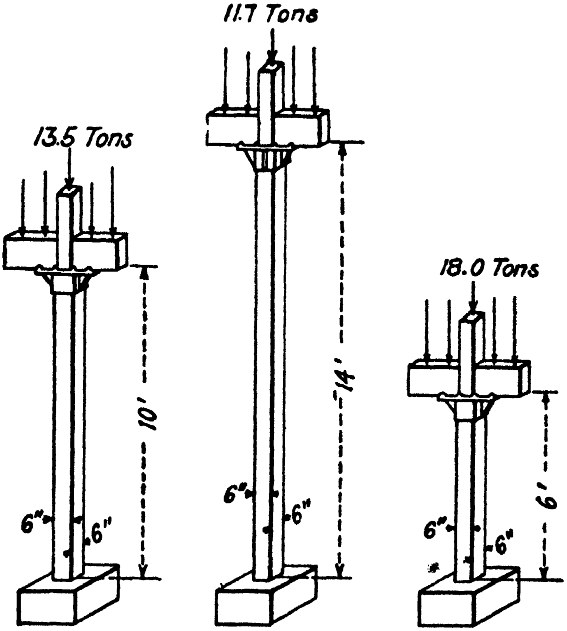
Since the quotient is less than 30, use Eq. (74). Substituting the values given,

$$p = 1{,}000\left(1 - \frac{120}{80 \times 6}\right)$$
$$p = 1{,}000(1 - \tfrac{120}{480}) = 1{,}000(1 - \tfrac{1}{4})$$
$$p = 1{,}000 \times \tfrac{3}{4} = 750 \text{ lb.}$$

But the total load $P = pA$.

$$\therefore P = 750 \times 36 = 27{,}000 \text{ lb., or } 13.5 \text{ tons}$$

In order to appreciate what effect the length of a column has on the total load carried, take a 6- by 6-in. column 14 ft. long, and find the safe load it will carry.



(a) (b) (c)

FIG. 423.

As before, $C = 1{,}000$ lb., $D = 6$, and $A = 36$ sq. in. But $L = 14 \times 12 = 168$ in. From Eq. (74),

$$p = 1{,}000\left(1 - \frac{168}{80 \times 6}\right) = 1{,}000\left(1 - \frac{168}{480}\right) = 650 \text{ lb.}$$

But

$$P = pA$$

$$\therefore P = \frac{650 \times 36}{2,000} = 11.7 \text{ tons}$$

This result shows that by an increase of the length from 10 to 14 ft. the load was decreased from 13.5 to 11.7 tons.

If this 6- by 6-in. timber were used as a short column, *i.e.*, one for which L divided by D does not exceed 10 or 12, the full compressive fiber stress of 1,000 lb. could be used. Then,

$$P = 1,000 \times A = 1,000 \times 36 = 36,000 \text{ lb.} \qquad P = 18 \text{ tons}$$

That is, if a 6- by 6-in. short block were used, it would safely carry 18 tons. The buckling effect is then quite evident as the length of the timber is increased.

Figures 423 *a*, *b*, and *c* show the columns in their relative lengths, with the loads that they will safely carry. In Fig. 423*c*, no buckling effect is considered. The student thus has a graphic illustration of the buckling effect on columns, *i.e.*, the effect of the length of column on the safe load it will carry after the length has passed a certain limit.

190. Selecting a Column When Its Length and the Load Are Given.—Suppose the builder must choose a wooden column 16 ft. long to carry a load of 100,000 lb. To find the cross-section area or size of the timber:

The required area depends on the compressive stress and on the ratio of the length of the column to its least dimension. In order that we may find the least possible dimension of the area, the column is treated as a short block. In that case, use a safe stress, say 1,000 lb. per sq. in., as if there were no tendency to buckle. The required area is found by dividing the total load by the fiber stress, or

$$A = \frac{W}{1,000} = \frac{100,000}{1,000} = 100 \text{ sq. in.}$$

But A being a square is equal to the product of its dimension, or $A = D \times D = D^2$.

$$\therefore D^2 = 100, \qquad D = 10 \text{ in.}$$

Then the least dimension must be 10 in. But when the column 16 ft. long is considered the allowable fiber stress is less and is found by Eq. (74), $D = 10$ and $L = 16 \times 12 = 192$ being used. Then

$$p = 1{,}000\left(1 - \frac{192}{80 \times 10}\right) = 1{,}000(1 - 0.24)$$

$p = 1{,}000 \times 0.76 = 760$ lb. per sq. in. This value is the allowable fiber stress for the column.

Since $W = 100{,}000$ lb. and $p = 760$, the required area is

$$A = \frac{W}{p} = \frac{100{,}000}{760} = 132$$

If the section of the column were a rectangle, a 10- by 14- (140-sq. in.) column would be satisfactory. If a square section is to be used, a 12- by 12-in. column would be required.

The student must remember that the value of p from Winslow's formula is the fiber stress which has the same effect on the column as has a fiber stress of 1,000, when the same timber is used as a short block.

191. Round Columns.—When a round column is used, D is the diameter of the cross section. Suppose the problem is to select a round 16-ft. column for the 100,000-lb. central load. As before, treat the timber as a short block, which can safely stand a compressive fiber stress of 1,000 lb. When there is no tendency to buckle, the area necessary is found from

$$A = \frac{W}{1{,}000} = \frac{100{,}000}{1{,}000} = 100 \text{ sq. in.}$$

But the section is a circle, and the area

$$A = 0.7854D^2 \qquad \therefore\ 0.7854D^2 = 100$$

$$D^2 = \frac{100}{0.7854} = 127$$

By extracting the square root,

$$D = 11.5.$$

Now, putting this value of D in Eq. (74), we have

$$p = 1{,}000\left(1 - \frac{16 \times 12}{80 \times 11.5}\right)$$

$$p = 1{,}000\left(1 - \frac{12}{57.5}\right) = 1{,}000(1 - 0.21), \qquad p = 790 \quad \text{lb.}$$

Since the load is 100,000 and the allowable fiber stress is 790, the necessary area is

$$A = \frac{100{,}000}{790} = 127$$

As before,

$$A = 0.7854D^2, \qquad \therefore 0.7854D^2 = 127$$

$$D^2 = \frac{127}{0.7854} = 160 \text{ approximately}$$

Since 13 × 13 = 169, a value of $D = 13$ for the diameter of the column is sufficient to carry the load.

As a check on the work, substitute $D = 13$ in Eq. (74).

$$p = 1{,}000\left(1 - \frac{L}{80D}\right) = 1{,}000\left(1 - \frac{192}{80 \times 13}\right)$$

$$p = 1{,}000(1 - 0.18) = 820 \text{ lb.}$$

This value is the allowable stress on a 13-in. column. But the actual stress is found by dividing the total load 100,000 by the area of the cross section or

$$\frac{100{,}000}{0.7854D^2} = \frac{100{,}000}{0.7854 \times 169} = 750 \text{ lb.}$$

But 750 lb. is less than 820 lb., and the design is on the side of safety. A little smaller column, say 12.5 in., could have been used.

192. Straight-line Formula.—In Art. 189, the statement was made that when the ratio of the length of a column to its least cross-sectional dimension was less than 10 the load it would carry was found by multiplying a safe fiber stress for the material by the cross-sectional area.

Experiment has shown that the safe working stress decreases as the slenderness ratio increases. The relation may be expressed as follows:

$$\frac{P}{A} = s - k\frac{l}{r} \tag{76}$$

where P is the load on the column, A its cross-sectional area, s the safe compressive fiber stress for a short block of the material, l the length of the column, r the least radius of gyration, and k a constant chosen from experience and experiment. Let

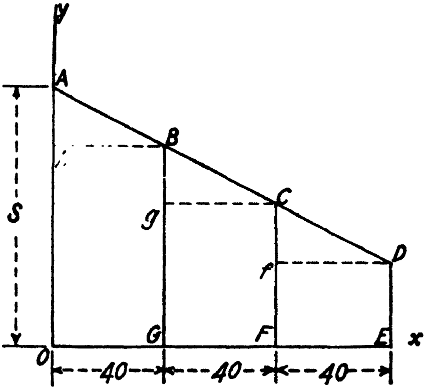


FIG. 424.

$$\frac{P}{A} = y, \qquad \text{and } \frac{l}{r} = x$$

Then the equation becomes

$$y = s - kx$$

When

$$\begin{cases} x = o \\ y = s \end{cases} \begin{cases} x = 40 \\ y = s - 40k \end{cases} \begin{cases} x = 80 \\ y = s - 80k \end{cases} \begin{cases} x = 120 \\ y = s - 120k \end{cases}$$

Plot these points as in Fig. 424.

$$hA = 40k, \qquad gB = 40k, \qquad fC = 40k$$

$$\frac{hA}{hB} = \frac{gB}{gC} = \frac{fC}{fD} = \frac{40k}{40} = k$$

Then triangles hAB, gBC, and fCD are similar, and A, B, C, and D are on a straight line.

Equation (76) is called a straight-line formula.

There are many straight-line formulas differing from each other in the values of s and k, with the limits of l/r within which the formula is valid carefully specified. Because of the uncertainty of the end conditions of the column and of the position of the center of the load, these formulas further specify an upper limit for P/A less than s when l/r is smaller than a specific value.

The Building Code Committee of the U. S. Department of Commerce in its 1926 report recommends

$$\frac{P}{A} = 18{,}000 - 70\frac{l}{r} \tag{77}$$

for steel that has passed a very rigid test, as specified in its report. For other grades of steel, it recommends

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{r} \tag{78}$$

for values of the slenderness ratio between 30 and 150. For values of l/r less than 30, use $P/A = 14{,}000$ lb.

The American Bridge Company specifies equation

$$\frac{P}{A} = 19{,}000 - 100\frac{l}{r} \tag{79}$$

for slenderness ratios between 60 and 120. For values of l/r less than 60, the maximum value of $P/A = 13{,}000$ is used. For values of l/r between 120 and 200, it recommends the use of the equation

$$\frac{P}{A} = 13{,}000 - 50\frac{l}{r} \tag{80}$$

The American Railway Engineering Association specifies the formula

$$\frac{P}{A} = 15{,}000 - 50\frac{l}{r} \tag{81}$$

for values of l/r between 50 and 100. For values less than 50 it specifies 12,500 as a maximum value for P/A. For cast-iron columns, Carnegie Steel Company specifies the formula

$$\frac{P}{A} = 9{,}000 - 40\frac{l}{r} \tag{82}$$

where P/A must not exceed 6,200 lb. which is the value when $l/r = 70$. In 1926, Eq. (82) was recommended by the Building Code Committee of the U. S. Department of Commerce for values of l/r as great as 90. If the value of P/A is reduced one-third, a slenderness ratio as great as 120 may be used.

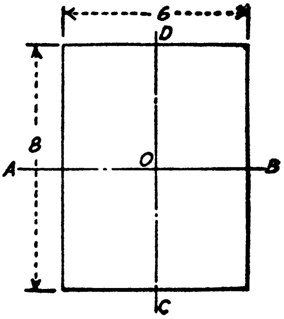


FIG. 425.

193. Radius of Gyration.—The radius of gyration of the cross section of any beam or column is found by dividing the moment of inertia I of the cross section by the area and extracting the square root of the quotient. If we represent the radius of gyration by r, then we may write the formula.

$$r = \sqrt{\frac{I}{A}}$$

Example.—Suppose, for example, that the cross section of a rectangular beam is 6 by 8 in., as in Fig. 425. Now, it has been shown that I for a rectangle about the gravity or neutral axis AB is

$$I = \tfrac{1}{12}bh^3$$

But

$$b = 6, \qquad \text{and } h = 8$$

$$\therefore I = \tfrac{1}{12} \times 6 \times 8 \times 8 \times 8 = 256$$

Also,

$$A = 6 \times 8 = 48 \text{ sq. in.}$$

$$\therefore r = \sqrt{\tfrac{256}{48}} = \sqrt{5.33} = 2.3 \text{ in.}$$

Again, if *CD* is taken as an axis through *O*, the center of gravity of the section, perpendicular to *AB*, then

$$b = 8, \qquad \text{and } h = 6$$

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12} \times 8 \times 6 \times 6 \times 6 = 144$$

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{144}{48}} = \sqrt{3} = 1.73$$

Now, it is proved in mechanics that if a figure has axes of symmetry or axes that divide the figure into two equal parts, such as *AB* and *CD*, the moment of inertia with respect to *AB* is greater than for any other axis through *O*, the center of gravity. Also, *I* for *CD* is less than for any other axis through *O*. In designing beams, the beam is strongest when placed so that the neutral axis, or axis of no stress, for a cross section is the gravity axis for which the moment of inertia is the greatest.

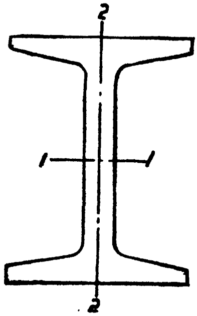


FIG. 426.

But in columns the moment of inertia used must be with respect to an axis for which it is the least, for it is about this axis that buckling of the column takes place, if the column fails under the load. But the least *I* is the one that gives the least radius of gyration *r*.

Now, for the different shapes of the cross sections of iron columns, it would be unnecessary work to calculate the radii of gyration, since they are given in handbooks. The next step is to give the student a few ideas in the use of a handbook.

Table XI (Appendix) contains some of the values in handbooks, on the detailed properties of I beams, a section that is shown in Fig. 426.

Figure 426 shows that axes $1-1$ and $2-2$ are axes of symmetry. Therefore, I_{1-1} is the greatest, and I_{2-2} the least for the gravity axes. Then in using an I beam as a column, the radius of gyration for axis $2-2$, which is seen to be the least, is the one to use.

The details of the structural members in the following examples may be taken from any structural steel handbook.

Example 1.—A 15-in. 42.9-lb. I beam is used as a 12-ft. column. Find the load it will carry by Eq. (79).

Solution.—The length $l = 12 \times 12 = 144$ in., least $r = 1.08$ and, $A = 12.49$. Then

$$\frac{l}{r} = \frac{144}{1.08} = 133$$

Since l/r is greater than 120, use Eq. (80), or

$$\frac{P}{12.49} = 13{,}000 - 50 \times 133$$
$$P = 12.49 \times 6{,}350 = 79{,}300 \text{ lb.}$$

Example 2.—By Eq. (78), find the load that a 15-in. 40-lb. channel will carry when used as a 6-ft. column.

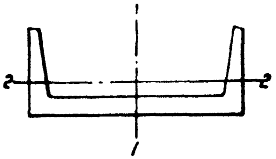


FIG. 427.

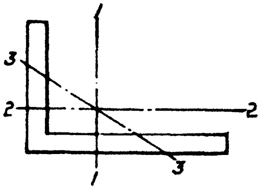


FIG. 427a.

Solution.—$l = 6 \times 12 = 72$, $A = 11.7$, and $r = 0.89$.

$$\frac{l}{r} = \frac{72}{0.89} = 81$$

Substituting in Eq. (78),

$$\frac{P}{11.7} = 16{,}000 - 70 \times 81$$
$$P = 11.7 \times 10.330 = 120{,}860 \text{ lb.}$$

Example 3.—Find the safe load that a 6- by 6- by 1-in. angle iron will carry when used as a column 15 ft. long.

Solution.—Among the elements given in the tables of steel handbooks are size, weight per foot, area of section, r for 1–1 axis, r for 2–2 axis, and r for 3–3 axis. The r's for axes 1–1 and 2–2 are used when angle irons are riveted to plates in built-up sections. When an angle iron is used alone as a column, it is always the r for axis 3–3 that is used, for it is the least.

$$l = 15 \times 12 = 180, \qquad r = 1.16, \qquad \text{and } A = 11$$

$$\frac{l}{r} = \frac{180}{1.16} = 155$$

Since l/r is greater than 120, use Eq. (80).

$$\frac{P}{11} = 13{,}000 - 50 \times 155$$
$$P = 11 \times 5{,}250 = 57{,}750 \text{ lb.}$$

Example 4.—By Eq. (80), find the safe load that a steel column 25 ft. long and 6 in. in diameter will carry.

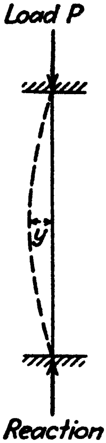


FIG. 428.

Solution.—The moment of inertia of a circle about its diameter is

$$\frac{\pi r^4}{4}$$

and

$$A = \pi r^2$$

Then

$$I = \frac{\pi 3^4}{4} = 20.25\pi, \qquad \text{and } A = \pi 3^2 = 9\pi$$
$$r = \sqrt{\frac{20.25\pi}{9\pi}} = 1.5$$
$$l = 25 \times 12 = 300$$
$$\frac{l}{r} = \frac{300}{1.5} = 200$$

Substituting in the formula,

$$\frac{P}{9\pi} = 13{,}000 - 50 \times 200$$
$$P = 9\pi \times 3{,}000 = 84{,}820 \text{ lb.}$$

The basis of column analysis and column design is given by Euler's formula

$$\frac{P}{A} = \frac{\pi^2 E}{(l/r)^2}$$

It will be accepted without any attempt at derivation, since the calculus is required. This formula is derived on the assumption that long columns would fail by bending sooner than they would by crushing. Thus, in Fig. 428, it can be seen that the load produces a bending effect. The deflection of the column as shown might represent either actual deflection or else the eccentricity of the load. Since a column could continue to bend without increase in load when the elastic limit is reached, the value of P in the formula is the ultimate, or limiting, load. The safe load would be obtained by applying a factor of safety. Thus, with a factor of safety of 2.0, the working formula is

$$\frac{P}{A} = \frac{\pi^2 E}{2(l/r)^2} \tag{83}$$

A curve drawn for a range of values of P/A and l/r is shown in Fig. 428*a*. From this curve, it is seen that for low values of l/r the values of P/A would be much in excess of the breaking strength of the material, whereas for large values of l/r the formula may be used to good advantage. This formula is seen to have decided limitations for use in designing columns of intermediate length.

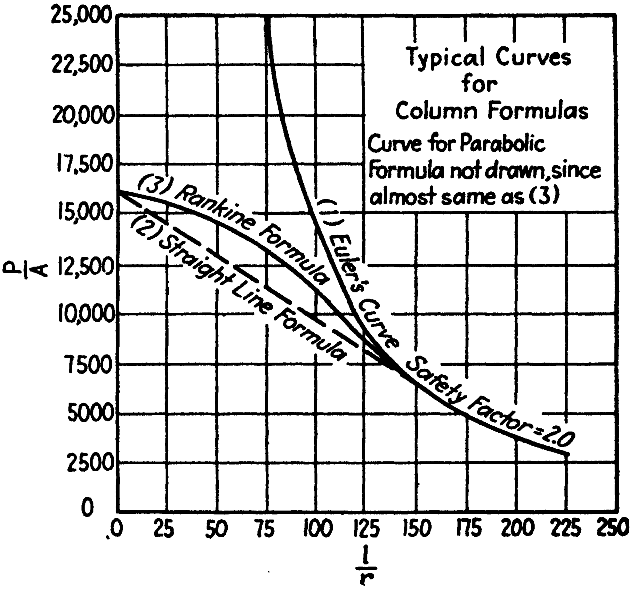


FIG. 428*a*.

In actual practice, most columns fall in the classification of intermediate length to which Euler's formula does not apply. However, by drawing a tangent to the curve and finding the equation of this line as shown in Art. 192, a suitable column formula is found which will cover usual practice. Thus a series of straight-line formulas have been derived which give safe results.

It can be seen from an inspection of Fig. 428*a* that the straight-line formula gives conservative values of the load. It was felt that some modification could be introduced. Accordingly, a curve has been drawn that lies above the straight line and gives values of load more nearly constant for short as well as intermediate columns. The formulas pertaining to these curves are of the Rankine type, of which a good illustration is the one recommended by the American Institute of Steel Construction. The results are in harmony with good design.

However, the Rankine type of equation is somewhat difficult to solve, and equations known as the *parabolic formulas* have come into use. The curves of these equations conform closely to the Rankine curves but have the advantage of a simpler formula of which a typical one is

$$\frac{P}{A} = 13{,}000 - 0.25\left(\frac{l}{r}\right)^2 \tag{84}$$

From the foregoing, it is seen that Euler's formula is used as a starting point for column theory and that various working formulas are obtained from the equations of curves drawn tangent to or intersecting Euler's curve at some particular point. These curves start from the elastic limit of the material as the maximum ordinate. In each case, to obtain a working formula, a factor of safety is applied.

The several equations involved in the development of practical column design are listed below. The last three are representatives of the types.

Euler's

$$\frac{P}{A} = \frac{\pi^2 E}{2.0(l/r)^2}$$

Straight-line

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{r}$$

Rankine's

$$\frac{P}{A} = \frac{18{,}000}{1 + \frac{1}{18{,}000}(l/r)^2}$$

Parabolic

$$\frac{P}{A} = 13{,}000 - 0.25\left(\frac{l}{r}\right)^2$$

This general explanation gives some reasons for the many formulas in existence and use.

194. Cast-iron Columns.—Figures 429, 430, and 431 show the shapes of the cross section of cast-iron columns in common use. For interior columns, the hollow cylindrical shape (Fig. 429) meets the requirements better and is more commonly used than any other shape. For exterior columns, as in store fronts, the shape shown in Fig. 430 is commonly used, for it gives a good bearing for the beams that support the walls above. In fireproof construction the H-shaped section (Fig. 431) is often used. This shape of column is readily protected by concrete and masonry, so that no open space is left which exposes the metal. The material of the column may be seen at all parts and is easily measured for uniform thickness.

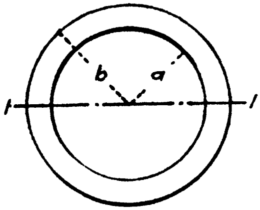


FIG. 429.

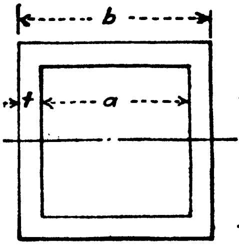


FIG. 430.

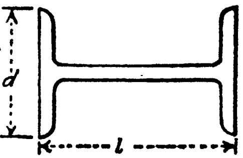


FIG. 431.

Although the safe load that a cast-iron column will carry is given in handbooks, examples will show how several are figured, as well as the use of the handbook in selecting a column to carry a given load.

Consider first the hollow cylindrical columns.

Since the dimension of the round cast-iron column is the same in every direction, all that is necessary is to calculate the radius of gyration about any axis, say 1 – 1 of Fig. 429.

Now the moment of inertia of a hollow circle is

$$I = \frac{\pi}{4}b^4 - \frac{\pi}{4}a^4$$

where b is the radius of the outer circle and a that of the inner one. Also,

$$A = \pi b^2 - \pi a^2$$

$$\therefore r = \sqrt{\frac{(\pi/4)b^4 - (\pi/4)a^4}{\pi b^2 - \pi a^2}} = \sqrt{\frac{b^2 + a^2}{4}} = \frac{1}{2}\sqrt{b^2 + a^2}$$

That is, to find r, simply square each radius, add the results, extract the square root of the sum, and divide by 2.

Thus, if $b = 6$ in. and $a = 4$ in.,

$$r = \frac{\sqrt{6^2 + 4^2}}{2} = \frac{\sqrt{52}}{2} = \frac{7.21}{2} = 3.6$$

Example 1.—Find the safe load that a 14-ft. round column will carry, if the outer radius is 4 in. and the metal 1 in. thick.

Solution.—In Fig. 429, $b = 4$, and $a = 3$. Then

$$r = \frac{\sqrt{4^2 + 3^2}}{2} = \frac{5}{2} = 2.5$$

$$A = \pi(4)^2 - \pi(3)^2 = 7\pi = 7 \times 3.1416 = 21.99 \text{ or } 22 \text{ approximately}$$

$$l = 14 \times 12 = 168 \text{ in.}$$

Substituting in Eq. (82),

$$\frac{P}{22} = 9{,}000 - 40 \times \frac{168}{2.5} = 9{,}000 - 2{,}690 = 6{,}310$$

$$\therefore P = 22 \times 6{,}310 = 139{,}000 \text{ lb.}$$

This result may be verified by consulting any recent structural steel handbook, where the load is given in thousands of pounds.

Example 2.—A hollow square cast-iron column is 20 ft. long, 10 by 10 in. in cross section, with metal 1 in. thick. Find the safe load that it would carry (Fig. 432).

Solution.—The advantages of using hollow square sections is that the radii of gyration about axes 1 – 1 or 2 – 2 are the same. The column is of equal strength with reference to either axis.

For hollow square sections,

$$r = \sqrt{\frac{b^2 + a^2}{12}}$$

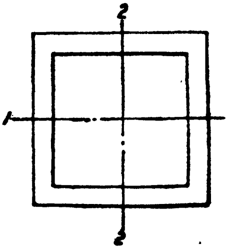


FIG. 432.

where b is the side of the large square and a the length of the side of the inner square. Notice that a is equal to b minus twice the thickness t of the metal, or

$$a = b - 2t$$

In the example, $b = 10$ and $a = 10 - 2 = 8$

$$r = \sqrt{\frac{(10)^2 + (8)^2}{12}} = 3.7$$

$$\frac{l}{r} = \frac{20 \times 12}{3.7} = 65, \qquad \text{and } A = (10)^2 - (8)^2 = 36$$

Substituting in formula (82),

$$\frac{P}{36} = 9{,}000 - 40 \times 65$$

$$P = 36 \times 6{,}400 = 230{,}400 \text{ lb.}$$

195. Alloy Columns.—The preceding formulas for steel columns apply to ordinary structural steel. Nickel steel and silicon steel have been used in highway and railway bridges. On account of the high yield point of these alloys, the formulas already given must be modified accordingly. The American Society of Civil Engineers specifications have included

$$\frac{P}{A} = \frac{24{,}000}{1 + \frac{1}{9{,}000}\left(\frac{l}{r}\right)^2} \tag{85}$$

The American Railway Engineering Association has used a parabolic type of formula for silicon steel columns with pin ends.

$$\frac{P}{A} = 20{,}000 - 0.61\left(\frac{l}{r}\right)^2 \tag{86}$$

For nickel steel with pin ends, this association gives

$$\frac{P}{A} = 22{,}500 - 0.76\left(\frac{l}{r}\right)^2$$

For aluminum alloys, recommended formulas are

$$\frac{P}{A} = 15{,}000 - 123\frac{l}{r}, \qquad \text{for } \frac{l}{r} \text{ to } 81 \qquad (88)$$

and

$$\frac{P}{A} = \frac{33{,}000{,}000}{(l/r)^2}, \qquad \text{for } \frac{l}{r} \text{ over } 81 \qquad (89)$$

Other formulas are in use and may be found in the specifications relating to the special project.

196. H-section Columns.—The figuring of the moment of inertia of an H section is not an easy problem. If one has access to a steel handbook, the least radius of gyration is easily obtained. However, if the handbook is not available, the following rule will be found to lead to values of r near enough for practical purposes. The least radius of gyration is equal to the quotient obtained by dividing the longest dimension or depth of the beam section of Fig. 431 by 4.2.

Example 1.—If the depth is 8 in., then

$$r = \frac{8}{4.2} = 1.9$$

A steel handbook gives $r = 1.87$.

Example 2.—Find the safe load that an H-shaped cast-iron column 10 ft. long, 8 in. deep, and weighing 34 lb. per ft. will carry.

Solution.—$l = 10 \times 12 = 120$ in., $r = 1.87$, and $A = 10$ (see any steel handbook). Substituting in

$$\frac{P}{A} = 9{,}000 - 40\frac{l}{r}$$

we have

$$\frac{P}{10} = 9{,}000 - 40 \times \frac{120}{1.87} = 9{,}000 - 2{,}560 = 6{,}440$$

$$\therefore P = 10 \times 6{,}440 = 64{,}400 \text{ lb.}$$

H sections are also made of steel. A complete table of safe loads, figured from the formula

$$\frac{P}{A} = 19{,}000 - 100\frac{l}{r}$$

may be found in steel handbooks.

Example 3.—Find the safe load that a 6-in. 24.1-lb. H-column will carry when the length is 22 ft.

Solution.—

$$r = \frac{6}{4.2} = 1.43 \text{ by the preceding rule.}$$

Also,

$$l = 22 \times 12 = 264 \text{ in.} \qquad \text{and } A = 7 \text{ sq. in.}$$

$$\frac{l}{r} = \frac{264}{1.43} = 185.$$

Since the slenderness ratio is greater than 120, use Eq. (80),

$$\frac{P}{7} = 13{,}000 - 50 \times 185$$

$$P = 7 \times 3{,}750 = 26{,}250 \text{ lb.}$$

197. Plate-and-channel Columns.—The handbooks published by large steel companies contain tables of safe loads for plate and channel columns of various lengths. With the parts arranged according to definite specifications, the designer has little occasion to calculate these values unless as a matter of check on the accuracy of the tables. However, in order that the student may be familiar with the methods employed in calculating the safe loads, a definite section will be taken as shown in steel handbooks and the computations shown.

Example 1.—Figure 433 shows the cross section of a column made by riveting two 15-in. channels to two 16- by $\frac{11}{16}$-in. plates. Find the safe load it will carry when 20 ft. long; when 30 ft. long.

Solution.—The figure also shows the details of one of the channels. For the column,

$$I_{1-1} = 2 \times 346.3 + 16 \times \tfrac{13}{16} \times 2(7.5 + \tfrac{13}{32})^2 = 2{,}315.3$$
$$A = 2 \times 11.7 + 2 \times \tfrac{13}{16} \times 16 = 49.4$$
$$r_{1-1} = \sqrt{\frac{2{,}315.3}{49.4}} = 6.85$$
$$I_{2-2} = 2 \times 9.3 + 2 \times 11.7(4.5 + 0.26)^2 + \tfrac{2}{12} \times \tfrac{13}{16} \times (16)^3$$
$$= 1{,}162.76$$
$$r_{2-2} = \sqrt{\frac{1{,}162.76}{49.4}} = 4.85$$

Then 4.85 is the least radius of gyration.

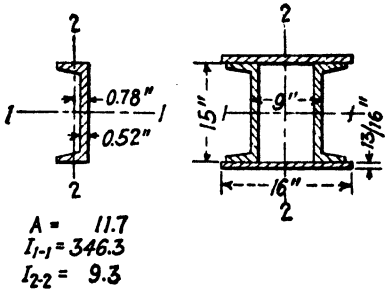


FIG. 433.

When l = 20 ft.,

$$\frac{l}{r} = \frac{240}{4.85} = 49 \text{ approximately}$$

Since l/r is less than 60, the allowable stress is 13,000. Then

$$P = 49.4 \times 13{,}000 = 642{,}000 \text{ lb.}$$

When l = 30 ft.,

$$\frac{l}{r} = \frac{360}{4.85} = 74 \text{ approximately}$$

Substituting in Eq. (79)

$$\frac{P}{49.4} = 19{,}000 - 100 \times 74$$
$$P = 49.4 \times 11{,}600 = 573{,}000 \text{ lb.}$$

In the handbook, the least r = 4.73. Allowance was undoubtedly made for the rivet holes, which was not considered in the foregoing calculation.

Example 2.—Find the safe load that a column made of two 12-in. 35-lb. channels and two 16- by $\frac{7}{16}$-in. plates will carry when 20 ft.

long; when 35 ft. long. The channels are placed 10 in. apart, back to back.

198. Rankine's Formula.—There are many types of the so-called "Rankine's formulas," but the only one mentioned in this work is the one recommended by the American Institute of Steel Construction, *viz.*,

$$\frac{P}{A} = \frac{18{,}000}{1 + \frac{1}{18{,}000}(l/r)^2} \tag{90}$$

In this formula, the symbols used have the same meaning as in the straight-line formulas, and l/r must not exceed 120 for main compression members or 200 for bracing or secondary members. Specifications frequently limit the maximum value of P/A to 15,000 lb. per sq. in. This value is obtained when l/r is about 60. With values of l/r less than 60, the column formula is not used. Tests seem to indicate that the stress at which failure occurs is little affected by values of l/r below 60. The formula makes no distinction between end conditions.

199. Derivation of Rankine's Formula.—In the derivation of Rankine's formula, it is assumed that the actual stress is a combination of a direct stress plus a stress due to bending. This assumption at once places the column to which the formula is to apply in the category of columns of intermediate length. Thus,

$$s = \frac{P}{A} + \frac{Pec}{I}$$

or since $I = Ar^2$

$$s = \frac{P}{A} + \frac{Pec}{Ar^2} = \frac{P}{A}\left(1 + \frac{ec}{r^2}\right)$$

Solving for P/A, we have

$$\frac{P}{A} = \frac{s}{1 + (ec/r^2)} \tag{q}$$

In this equation, e denotes the moment arm of the load with respect to the center of the cross section farthest removed from the line of action of the load. Thus, e might be a value of eccentricity of the load, or if the column is bent it will represent the deflection of the column.

It is assumed that within the elastic limit there is a similarity between the action of the beam and the column. For the beam the bending moment relation may be expressed as

$$KWl = \frac{pI}{c}$$

and the deflection relation as

$$kWl^3 = y$$

Eliminating W between these two equations produces

$$\frac{pI}{Klc} = \frac{y}{kl^3}$$

Then

$$yc = \frac{PIkl^3}{Kl} = \left(\frac{PIk}{K}\right)l^2$$

Or

$$yc = Cl^2$$

For the beam, yc is the product of the deflection times the distance to the extreme fiber. For the column, ec is the product of the deflection of the column by the distance to the extreme fiber. Thus, by analogy, since $yc = Cl^2$ for the beam, it seems logical to write $ec = ql^2$ for the column.

Equation (q) may be written

$$\frac{P}{A} = \frac{s}{1 + (ql^2/r^2)} \qquad (r)$$

In the American Institute of Steel Construction formula, $q = 1/18{,}000$ and s is given as 18,000. The values of the constant q are determined by experiment for various end conditions.

Example.—Solve Example 1, Art. 197, by the Rankine formula.

Solution.

$$\frac{P}{49.4} = \frac{18,000}{1 + \frac{(360)^2}{18,000(4.85)^2}}$$

$$P = 49.4 \times 13,780 = 680,730 \text{ lb.}$$

200. Selecting a Column.—The problem of designing a column is really the problem of selecting a cross section in accordance with given specifications and some suitable formula. This problem is more difficult than finding the load that a given section will carry, since neither A nor r is known. Usually there is no simple relation between A and r for the rolled sections that are commonly used. Consequently, a direct solution is not possible, and the method of cut and try is employed, which is found entirely satisfactory.

Example 1.—By means of the formula $P/A = 18,000 - 70(l/r)$, find the diameter of the steel column 15 ft. long, to carry a load of 200,000 lb.

Solution.—For a circle of radius a

$$r = \sqrt{\frac{\pi a^4}{4} \div \pi a^2} = \frac{a}{2}$$

Substituting in the formula,

$$\frac{200,000}{\pi a^2} = 18,000 - 70\frac{180 \times 2}{a}$$

Clearing of fractions and simplifying,

$$a^2 - 1.4a - 3.54 = 0$$

Solving the quadratic,

$$a = 2.7 \text{ in.,} \qquad \text{or } d = 5.4 \text{ in.}$$

Example 2.—Find the minimum size for a 15-ft. hollow square column to carry 175,000 lb.

Solution.—The greatest length of the column is to be taken as seventy times the least radius of gyration.

But

$$l = 15 \times 12 = 180 \text{ in.}$$

Then

$$180 = 70 \times r$$
$$r = 2.57$$

Also, for greatest length of columns the maximum fiber stress is 6,200 lb. The required area is found by dividing the load by 6,200, or

$$A = \frac{175,000}{6,200} = 28.2 \text{ sq. in.}$$

The nearest commercial column having a value of A greater than 28.2 is one with an area of 30.94, with $r = 2.84$ for a square section 8 in. on the side and a metal thickness of $t = 1\frac{1}{8}$ in. This column is larger than necessary, but it is the nearest commercial size.

Example 3.—Select a typical plate-and-channel column, 35 ft. long, to carry a load of 500,000 lb. by means of the formula $P/A = 16,000 - 70(l/r)$.

Solution.—From the Carnegie handbook a column made up of two 12-in. 35-lb. channels and two 14- by $1\frac{3}{8}$-in. plates will carry 512,000 lb. For this channel,

$$A = 59.02, \qquad r = 4.07, \qquad \frac{l}{r} = \frac{420}{4.07} = 103$$

The formula is then valid, for the slenderness ratio is less than 120.

$$\frac{P}{A} = \frac{500,000}{59.02} = 8,475 \text{ lb.}$$

This value is the developed stress. The allowable stress is

$$16,000 - 70 \times 103 = 8,790 \text{ lb./sq. in.}$$

Since the allowable stress is greater than the developed stress, the column described above will carry the load safely.

201. The Forest Products Laboratory Formula.

—The most recent and reliable formula for the design of wooden columns of intermediate lengths is

$$\frac{P}{A} = S\left[1 - \frac{1}{3}\left(\frac{l}{Kd}\right)^4\right] \qquad (91)$$

where P, A, and l have the same meaning as in the previous column formulas, d is the least cross-sectional dimension of the column, and K is a constant depending on the grade and kind of timber, as well as the conditions under which it is used. Table XVII (Appendix) contains values of K and S to be used for the various cases. S represents the allowable compressive stress for short blocks parallel to the grain.

This formula is the one recommended by the American Society for Testing Materials and the American Railway Engineering Association in cooperation with the Forest Products Laboratory of the U. S. Forest Service. It has been adopted by the Building Code Committee of the U. S. Department of Commerce and by the National Lumber Manufacturers Association. A complete discussion of the findings and recommendations of these societies is found in Part II of the 1927 Standards of the American Society for Testing Materials.

Among the limitations in the use of Eq. (91) are the following:

1. The unit working stresses in compression parallel to grain for columns whose ratio of unsupported length to least dimension does not exceed 10 shall be not greater than that given for the species in Table XVII (Appendix) on working stresses.

2. For columns of which the ratio of unsupported length to least dimension is greater than 10, the Forest Products Laboratory formula shall be used until the reduction in allowable stress equals one-third the stress for short columns.

Substituting $P/A = \frac{2}{3}S$ in Eq. (91),

$$\frac{2}{3}S = S\left[1 - \frac{1}{3}\left(\frac{l}{Kd}\right)^4\right].$$

Simplifying and solving,

$$\frac{l}{d} = K\,.$$

That is, for a column of select Douglas fir for which $K = 23.7$, the maximum value of l/d is 23.7. However, from Table XVII (Appendix), it is evident that an average value of K would be approximately 25. In the following problems the assumption is made that l/d does not exceed 25. For a more exact solution, the actual values of K may be used. For columns of greater length, the committee recommends the following formula:

$$\frac{P}{A} = \frac{\pi^2}{36}\frac{E}{(l/d)^2} \tag{92}$$

in which a factor of safety of 3 is used. Since the specifications further limit column formulas to l/d not greater than 50, the general statement can then be made as follows:

For l/d between 0 and 25, use Eq. (91); for l/d between 25 and 50, use Eq. (92).

Example 1.—Find the safe load that a 6- by 6-in. square select southern-pine column will carry when 8 ft. long; when 12 ft. long; when 14 ft. long.

Solution.—From Table XVII (Appendix),

$$S = 1{,}600 \text{ lb./sq. in.}$$
$$K = 23.7$$

Substituting in Eq. (91),

$$P = 36 \times 1{,}600\left[1 - \frac{1}{3}\left(\frac{8 \times 12}{23.7 \times 6}\right)^4\right]$$
$$P = 53{,}600 \text{ lb.}$$

Since the formula is limited to $l/d = 25$, the greatest length for which this timber may be used as a column is $l = 25 \times 6 = 150$ in., or 12.5 ft. Therefore, figured by this formula a 6- by 6-in. timber should not be used as a column 14 ft. long.

The load for the 12-ft. length is left as an exercise for the student to compute.

For the load that the 14-ft. length will carry, substitute in Eq. (92). Then

$$\frac{P}{6 \times 6} = \left(\frac{9.87}{36}\right)\frac{1{,}600{,}000}{\left(\frac{14 \times 12}{6}\right)^2}$$
$$P = 20{,}140 \text{ lb.}$$

Example 2.—Find the size of a square column of Douglas fir, dense select, 20 ft. long, to carry a load of 100,000 lb.

Solution.—From Table XVII (Appendix), $S = 1{,}750$ lb. per sq. in. and $K = 22.6$. Let d be the dimension of the cross section. From the formula,

$$\frac{100{,}000}{d^2} = 1{,}750\left[1 - \frac{1}{3}\left(\frac{240}{22.6d}\right)^4\right]$$

Simplifying and clearing of fractions,

$$\begin{aligned} d^4 - 57d^2 - 4{,}210 &= 0 \\ d^2 &= 99.8 \\ d &= 10 \text{ in.} \end{aligned}$$

Therefore, a 10- by 10-in. column will safely carry the load. Since $l = 240$ and $d = 10$ in., l/d is less than 25, and the solution is valid.

Example 3.—Find the size of a square southern-pine dense select timber to be used as a column 25 ft. long to carry a load of 75,000 lb.

Solution.—From Table XVII (Appendix), $S = 1{,}750$, and $K = 22.6$. If the column were a short block, the allowable fiber stress would be 1,750 lb. per sq. in. The necessary area

$$\begin{aligned} A &= \frac{75{,}000}{1{,}750} = 42.8 \text{ sq. in.} \\ A &= d^2 = 42.8 \\ d &= 6.5 \text{ in.} \end{aligned}$$

These values are for the smallest timber that could possibly be used if there were no column action. The required cross section must be greater. Since l/d must not exceed 25, the least value of $d = (25 \times 12)/25 = 12$ in. Try a 12- by 12-in. timber.

$$\frac{P}{A} = \frac{75{,}000}{12 \times 12} = 521 \text{ lb./sq. in.}$$

This value is the developed stress.

$$1{,}750\left[1 - \frac{1}{3}\left(\frac{25 \times 12}{22.6 \times 12}\right)^4\right] = 893 \text{ lb./sq. in.}$$

This value is the allowable stress. Since the allowable stress is greater than the developed, the 12- by 12-in. timber will carry the load. Any greater cross section would be valid. This method is a cut-and-try method for choosing a commercial size.

Example 4.—Solve Example 3 by the Winslow formula.

Solution.—$C = 1{,}750$ lb. per sq. in., and l/d must not exceed 30. Then least $d = 10$ in.

$$\frac{P}{A} = \frac{75{,}000}{100} = 750 \text{ lb./sq. in.}$$

$$1{,}750\left(1 - \frac{300}{80 \times 10}\right) = 1{,}094 \text{ lb./sq. in.}$$

Since 750 < 1,094, the 10- by 10-in. timber would carry the load.

Example 5.—Solve Example 3 by the exact method used in Example 1.

Solution.—The solution gives $d = 11.7$ in., approximately. Therefore, use the 12- by 12-in. timber as found before.

Example 6.—Find the size of a square Douglas-fir coast-region select timber used as a column 20 ft. long, to carry 160,000 lb. $S = 1{,}600$ lb./sq. in., and $K = 23.7$.

202. Reinforced-concrete Columns.—Reinforced-concrete columns are in reality piers, and are not designed by the formula for the usual column type. Building codes generally specify the ratio of the length l to the least cross-sectional dimension d. The maximum value of l/d is usually limited to 15. With such a low ratio, it must be evident that the concrete column classifies as a short compression member or pier.

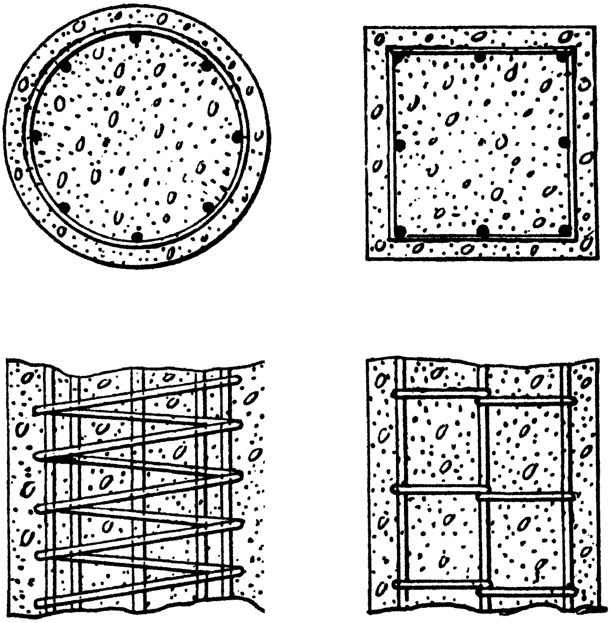


FIG. 434.

Concrete columns are reinforced with longitudinal bars placed about 2 in. from the surface of the column. The longitudinal bars are frequently enclosed within a spiral or hooped reinforcement and in square columns are connected by means of ties. The object of the ties and hoops is to keep the bars from buckling.

The percentage of longitudinal steel is assumed at 0.5 to 5 per cent. A large percentage is uneconomical; for the steel stress is low, and a value of 2 per cent of steel or less is desirable. Figure 434 shows cross sections of concrete columns and part elevations in which the longitudinal and hooping steel are indicated.

The allowable load that a reinforced-concrete column will support is usually assumed to be equal to the supporting value of the concrete plus the supporting value of the longitudinal steel. Since the steel and the concrete must act together as a unit, these materials must suffer the same unit deformation and the stresses will be proportional to the moduli of elasticity of the steel and concrete.

If Δ is the deformation, then

$$\Delta = \frac{s_s}{E_s}, \qquad \Delta = \frac{s_c}{E_c}, \qquad \text{or } \frac{s_s}{E_s} = \frac{s_c}{E_c}$$

Then

$$s_s = \frac{s_c E_s}{E_c}, \qquad \text{or } s_s = n s_c$$

The allowable load may be represented in terms of the following equation:

$$P = A_c s_c + A_s s_s$$

A_c is the effective area of the concrete, and A_s is the area of the steel. The term $A_c s_c$ is the resistance of the concrete, and $A_s s_s$ is the resistance of the steel.

If A represents the total cross-sectional area of the column, then A_s may be written as pA and $A_c = A - pA$. Taking advantage of these relations, we may write the equation for the total load

$$P = (A - pA)s_c + pAns_c$$

Taking out the common factor s_cA,

$$P = s_cA(1 - p + pn)$$

which may be further rearranged as

$$P = s_cA[1 + p(n - 1)] \tag{93}$$

Example 1.—A concrete column is 20 in. in diameter and 16 ft. long. It is reinforced with eight longitudinal bars each 1 in. in diameter. If s_c is limited to 650 lb. per sq. in., what will be the allowable load P? Assume $n = 15$.

Solution.

$$\begin{aligned} A &= \pi(10)^2 = 314.16 \text{ sq. in.} \\ A_s &= 8 \times \pi(\tfrac{1}{2})^2 = 2\pi \\ p &= \frac{A_s}{A} = \frac{2\pi}{314.16} = 0.02 = 2 \text{ per cent} \\ P &= 650 \times 314.16[1 + 0.02(15 - 1)] = 261{,}000 \text{ lb.} \end{aligned}$$

In the solution, the diameter of 20 in. was assumed to be the effective diameter. About 2 in. of concrete will be needed to protect the steel, and the total diameter of the column will be 24 in. This outer 2-in. shell is not considered as a part of the compression area.

Example 2.—Circular concrete columns 14 ft. high are placed at the corners of 18- by 18-ft. floor panels. The total dead and live load to be supported is 400 lb. per sq. ft. What should be the size of the columns?

$$s_c = 600 \text{ lb./sq. in.}, \qquad n = 15$$

Solution.—No percentage of steel has been given, and so this will be assumed at 2 per cent. Then,

$$P = 400 \times 18 \times 18 = 129{,}600 \text{ lb}$$

Substituting in the formula for P,

$$129{,}600 = A \times 600[1 + 0.02(15 - 1)]$$

Then

$$\begin{aligned} A &= \frac{129{,}600}{600 \times 1.28} = 168.5 \text{ sq. in.} \\ A &= \pi r^2 = 168.5 \\ r &= \sqrt{\frac{168.5}{\pi}} = 7.34 \text{ in.} \\ d &= 2r = 2 \times 7.34 = 14.68 \text{ in.} \end{aligned}$$

This gives the effective diameter. A covering of 2 in. to protect the steel will be added, so that the diameter is increased by 4 in., and will be 18.68 in.

203. Eccentric Load on Long Columns.—When a load is eccentric, in addition to the compressive and buckling stresses there will be a bending stress. From (1) the stress developed by compression is P/A. From Art. 168, the fiber stress due to bending from an eccentric load is Pac/I in which a is the eccentricity. But $I = Ar^2$ where r is the radius of gyration. Then the developed stress in the column is

$$\frac{P}{A} + \frac{Pac}{Ar^2} = \frac{P}{A}\left(1 + \frac{ac}{r^2}\right) \tag{94}$$

This value of the developed stress being equated to the allowable stress, in the straight-line formulas it becomes

$$\frac{P}{A}\left(1 + \frac{ac}{r^2}\right) = C\left(1 - \frac{L}{80D}\right), \text{ for wood}$$

$$\frac{P}{A}\left(1 + \frac{ac}{r^2}\right) = 16{,}000 - 70\frac{l}{r}, \text{ for steel}$$

$$\frac{P}{A}\left(1 + \frac{ac}{r^2}\right) = 19{,}000 - 100\frac{l}{r}, \text{ for steel}$$

$$\frac{P}{A}\left(1 + \frac{ac}{r^2}\right) = 18{,}000 - 70\frac{l}{r}, \text{ for steel}$$

$$\frac{P}{A}\left(1 + \frac{ac}{r^2}\right) = 9{,}000 - 40\frac{l}{r}, \text{ for cast iron}$$

$$\frac{P}{A}\left(1 + \frac{ac}{r^2}\right) = S\left[1 - \frac{1}{3}\left(\frac{l}{kd}\right)^4\right] \text{ for wood}$$

Example 1.—Find the load that an 8- by 8-in. timber, 20 ft. long, will carry when the load is central; eccentric 1 in.; eccentric 2 in.

Solution.—For the central load, use Eq. (74).

$$\frac{P}{64} = 1{,}000\left(1 - \frac{20 \times 12}{80 \times 8}\right)$$

$$P = 64{,}000(1 - \tfrac{3}{8}) = 40{,}000 \text{ lb.}$$

For the second case, r^2 must be calculated.

$$r^2 = \frac{I}{A} = \frac{\frac{1}{12} \times 8^4}{8^2} = \frac{16}{3}$$

Also, $a = 1$, $c = 4$, $L = 240$, and $D = 8$.

$$\therefore \frac{P}{64}\left(1 + \frac{1 \times 4}{\frac{16}{3}}\right) = 1{,}000\left(1 - \frac{20 \times 12}{80 \times 8}\right)$$

$$\frac{P}{64} \cdot \frac{7}{4} = 625$$

$$P = 23{,}000 \text{ lb. approximately}$$

For $a = 2$,

$$\frac{P}{64}\left(1 + \frac{2 \times 4}{\frac{16}{3}}\right) = 625$$

$$P = 16{,}000 \text{ lb.}$$

204. Compression Flange as a Column.—In the previous discussions of beams, no mention has been made of the fact that the compression flange of a beam or girder behaves like a column. It differs from true-column action owing to the fact that in the beam the compression varies from a maximum at the outer fiber to a minimum near the web and also that the stress varies from a maximum at some section to zero at some other section. Because of these variations, larger values of unit compression may be permitted than would be the case in true-column action. The situation in general is taken care of by supporting the compression flange laterally at intervals or else reducing the allowable compressive stress by means of some column formula.

A formula specified by the American Institute of Steel Construction is

$$\frac{P}{A} = \frac{20{,}000}{1 + \frac{1}{2{,}000}\left(\frac{l}{b}\right)^2}$$

The length l is the unsupported length, and b is the width of the compression flange. Good practice seems to indicate that l/b should not be much in excess of 15. If the value of l/b is below 10, no reduction in the stress will be needed.

The U. S. Department of Commerce recommends the straight-line formula

$$\frac{P}{A} = 22{,}000 - 270\frac{l}{b}$$

The formula specified by the American Bridge Company is

$$\frac{P}{A} = 19{,}000 - 300\frac{l}{b}$$

Example.—What is the allowable bending stress in the compression flange of an 18-in. 54.7-lb. I beam, used as a simple beam 20 ft. long and supporting a uniform load. There are no lateral supports. Use the American Bridge Company formula.

Solution.

$$l = 20 \times 12 = 240 \text{ in.,} \qquad b = 6 \text{ in.}$$

$$s_c = \frac{P}{A} = 19{,}000 - \frac{300 \times 240}{6} = 7{,}000 \text{ lb./sq. in.}$$

If a compressive stress of 16,000 lb./sq. in. is to be developed, then lateral supports must be provided. The equation will be solved for l. Then

$$16{,}000 = 19{,}000 - \frac{300l}{6}, \qquad l = 60 \text{ in.}$$

The supports should be 60 in. apart, five in all being required.

205. Column Action in the Web of an I Beam.—At any point in a beam, vertical and horizontal shearing stresses of the same intensity are produced (Fig. 435). It has been shown that these shearing stresses produce a compressive stress at the same intensity as the shearing stress and at an angle of 45° with the vertical. The web of an I beam may be considered as made up of parallel diagonal columns fixed at the ends by the flanges of the beam. The column has a thickness equal to that of the web, and its length is equal to the unsupported depth of the web divided by cos 45°.

The unit shearing stress in the I beam is determined from

$$s_h = \frac{V}{td}$$

This shearing stress is equal to the unit compressive stress. By means of some column formula, the allowable compressive stress may be determined, and the above formula may be written as

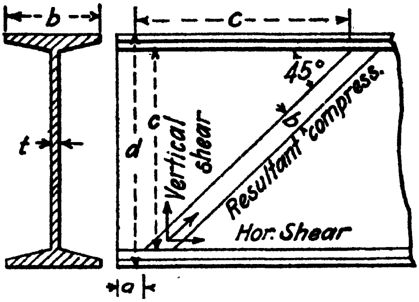


FIG. 435.

$$s_h = s_c = \frac{V}{td}$$

$$V = s_c td$$

When the total shear is known, the reaction may be computed, and the total safe load on the beam is determined. The following is one of several formulas recommended for use:

$$s_c = \frac{P}{A} = \frac{12{,}000}{1 + \frac{1}{36{,}000}\left(\frac{l}{r}\right)^2}$$

From Fig. 435, it can be seen that l^2 is equal to $2c^2$, where c is the unsupported depth of the web. The least moment of inertia may be written as $\mathrm{I} = bt^3/12$. The area of the column is $A = bt$, and the square of the radius of gyration is

$$r^2 = \frac{\mathrm{I}}{A} = \frac{bt^3}{12} \div bt = \frac{t^2}{12}$$

Making these substitutions in the column formula above produces

$$s_c = \frac{P}{A} = \frac{12{,}000}{1 + \frac{1}{36{,}000}\left(\frac{2c^2}{t^2/12}\right)} = \frac{12{,}000}{1 + \frac{1}{1{,}500}\left(\frac{c}{t}\right)^2}$$

Example.—What is the maximum uniform load that a 24-in. 79.9-lb. I beam can support if used as a simple beam?

Solution.—From a steel handbook, c, the unsupported depth of web is 21.72 in. The value of t = 0.5 in. Then

$$s_c = \frac{12{,}000}{1 + \frac{1}{1{,}500}\left(\frac{21.72}{0.5}\right)^2} = 5{,}330 \text{ lb./sq. in.}$$

$$V = 5{,}330 \times 24 \times 0.5 = 63{,}960 \text{ lb.}$$

For a simple beam,

$$W = 2V = 2R = 63{,}960 = 127{,}920 \text{ lb.}$$

Problems

359. Find the maximum lengths for which southern-pine select timbers of the following dimensions may be used: 4 by 4 in., 6 by 6 in., 8 by 8 in. What will be the central load in each case (Table XVII, Appendix)?

360. Find the central loads that 12-in. 40-lb. I beams will support if used as columns with lengths, respectively, 10 ft., 18 ft., 25 ft. How do these results compare with the safe loads given in the handbooks?

361. What must be the dimensions of a square oak column 12 ft. long to carry a central load of 15 tons?

362. A southern dense pine column 16 ft. long is to carry a central load of 120,000 lb. What should be the dimensions?

363. How far apart should 4- by 4-in. oak posts be placed, if 8 ft. in length, to support a platform with a floor load of 200 lb. per sq. ft.?

364. Find the safe load that a column 24 ft. long, consisting of a 12- by ½-in. web plate, four 6- by 4- by ½-in. angle irons, and two 14- by ½-in. cover plates, will support. Short legs of the angles are to be riveted to the web plate.

365. A hollow round cast-iron column 20 ft. long is to carry a load of 250 tons. If the external radius is 7.5 in., what is the internal radius?

366. Find the smallest 16-ft. hollow square cast-iron column to carry 150,000 lb.

367. Find the load that an 8-in. 34-lb. H column will carry if the length is 14 ft.

368. Find the dimensions of a square steel strut 10 ft. long to carry 20,000 lb.

369. Choose the proper I beam for a 15-ft. column to carry a central load of 200,000 lb.

370. A round solid cast-iron column 15 ft. long is to carry a load of 20 tons. Find the diameter.

371. Find what load a solid round steel column will carry if it is 6 in. in diameter and 12 ft. long, with the load 1 in. off center.

372. Find the size of a rectangular column 18 ft. long to carry a load of 20 tons acting 3 in. off center.

373. A floor is supported by reinforced-concrete columns spaced 15 ft. on centers in each direction. What size of columns will be needed to support a total dead and live load of 600 lb. per sq. ft. of floor area. Let $s_c = 650$, $n = 15$.

374. A concrete column 12 ft. long and 16 in. square is reinforced with eight ¾-in. steel rods. The outer surfaces of the rods are 1 in. from the nearest faces of the column. If $n = 10$ and $s_c = 600$, what safe load will this column carry?

375. The outside diameter of a concrete column is to be 15 in. The reinforcing bars must be covered with 1½ in. of concrete. If $n = 15$ and $s_c = 750$ lb. per sq. in., how much longitudinal steel is needed if the load on the column is 120,000 lb.?

376. A square concrete column is 16 ft. high and supports a load of 125,000 lb. Let $s_c = 500$ lb. per sq. in. and $n = 12$, with $p = 2$ per cent. What must be the outside dimensions of the column if 2 in. of concrete is used as a covering for the steel?

377. A concrete column 24 in. in diameter and 18 ft. high is reinforced with eight longitudinal rods each 1 in. square. What will be the allowable load if $s_c = 600$ lb. per sq. in. and $n = 15$? Assume 20 in. as the effective diameter.

378. What will be the allowable bending stress for a 9-in. 25-lb. I beam 16 ft. long with no lateral supports? What will be the spacing of lateral supports if a compressive stress of 12,500 lb. per sq. in. is to be developed?

379. In the problem above, if lateral supports are placed at 4-ft. intervals, what will be the allowable stress in compression? What will be the total load on the beam?

380. A 6-in. 12.5-lb. I beam 14 ft. long is supported at the ends. It carries a uniformly distributed load. There are lateral supports at the ends and in the middle. What is the allowable load on the beam?

381. What is the maximum end reaction for a 12-in. 35-lb. I beam used as a simple beam, if buckling of the web is to be avoided?

382. A 15-in. 65-lb. I beam is used in a simple span. What is the minimum length in which the full bending stress of 16,000 lb. per sq. in. may be developed without buckling the web?

383. Find the safe uniformly distributed load that a 10-in. 25.4-lb. I beam will support if the simple span is 12 ft.?

384. Select an I beam to carry a distributed load of 1,200 lb. per ft. for a simple span of 16 ft. It is supported laterally at the ends.

385. A simple beam with a span of 18 ft. is to carry a uniformly distributed load of 500 lb. per ft. and a concentrated load of 3,000 lb. 6 ft. from one end. If the beam is supported laterally at both ends, what size of I beam will be suitable?

386. A wide flange beam 10 ft. long is to be used as a column. If a 14- by 16-in. beam weighing 150 lb. per ft. is selected, what safe load will this column carry? Use formulas (78) and (84).

387. A column is to support a load of 269,000 lb. A wide flange beam 12 ft. long is to be used. Make a suitable selection. Use formula (90).

388. What would be a suitable selection of a wide flange beam to satisfy the requirements of Prob. 369? Use formula (90).

CHAPTER XVIII

TORSION, SHAFTS AND COUPLINGS

206. Torsion.—In Fig. 436, T_1 and T_2 are the tensions on a belt with $T_2 > T_1$. The difference in tensions, together with the horizontal reactions of the bearings, forms a couple tending to rotate the shaft counterclockwise. A rope is wound around the shaft as shown, with a weight W attached to the rope. This weight, together with the vertical reactions, forms a couple tending to rotate in a clockwise direction. If friction of bearings is neglected and the shaft is stationary or rotating uniformly, these couples have equal numerical moments, but opposite in sign. When a shaft is acted upon by two couples in parallel planes, the couples having equal numerical moments, but opposite in sign, it is said to be in *torsion.*

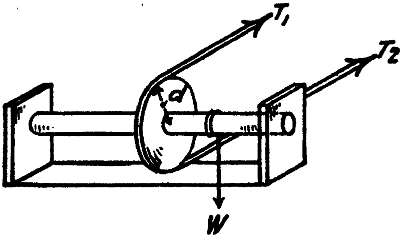


FIG. 436.

The moment or torque of either of the couples is called the *twisting moment*. It is generally expressed in inch-pounds.

If, in Fig. 436, $T_2 = 250$, $T_1 = 100$, and $d = 6$ in., the twisting moment is $(250 - 100) \times 6 = 900$ in.-lb.

If the radius of the shaft is 1 in., then if the couples are balanced, W must equal 900 lb., for its moment must also be 900 in.-lb.

In the previous illustration the turning moment or torque is constant throughout the entire length of the shaft. However, if a shaft has several pulleys attached to it, the torque will be different in different sections of the shaft.

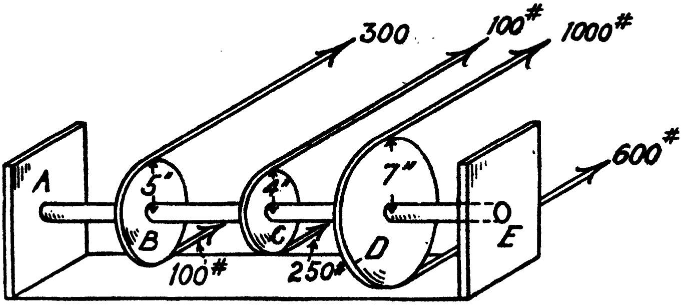


FIG. 437.

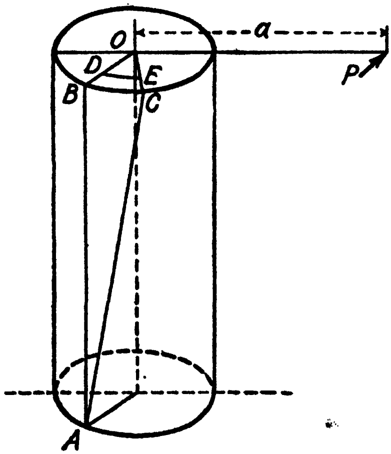


FIG. 438.

In Fig. 437, from E to D there is a negative, or clockwise, torque of $400 \times 7 = -2{,}800$ in.-lb.; from C to D, $-400 \times 7 + 150 \times 4 = -2{,}200$; from B to C, $-400 \times 7 + 150 + 4 - 200 \times 5 = -3{,}200$ in.-lb.

When two pipe wrenches are applied to a pipe on opposite sides of a joint for the purpose of unscrewing the pipes, the pipes are in torsion.

207. Shearing Stress in Terms of Torque.—When a shaft is in torsion, any cross section of the shaft tends to slip by or shear across the adjacent face. The resistance of the fibers of the shaft is called the *shearing stress* due to torsion. In Fig. 438, when P is applied to the lever a, an element AB, originally straight, is twisted into the position AC. A radius OB of the end is rotated into the position OC. Any portion of material originally at D is elongated to E. Now, the unit stress resisting that change is proportional to the unit elongation. But the elongation DE is proportional to the distance OD from the center of the shaft. Then, within the elastic limit of the material, the unit shearing stress at any point of the cross section of a shaft is proportional to the distance of that point from the center of the section.

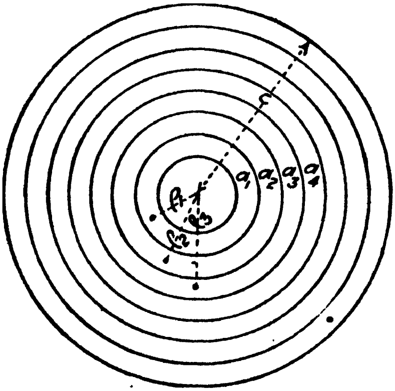


FIG. 439.

The center of the section has no stress. Then the axis of the shaft is a neutral axis. If the radius OB is 6 in. and the unit stress at B is 6,000 lb. per sq. in., then if D is 3 in. from the center, the stress at D is 3,000 lb.; if D is 1 in. from the center O, the stress is 1,000 lb.; etc.

An enlarged cross section of the shaft of Fig. 438 is shown in Fig. 439. A series of concentric circles are drawn, dividing the large circle into rings. Let $a_1, a_2, \ldots, a_8$ be the area of these successive rings, and $\rho_1, \rho_2, \ldots, \rho_8$ be the mean, or average, radii for the rings $a_1, a_2, \ldots, a_8$, respectively. Also let $p_1, p_2, \ldots, p_8$ be the average unit shearing stress on the areas $a_1, a_2, \ldots, a_8$, respectively. Then the total shearing stress on a_1 is a_1p_1. The torque or moment of this shearing force is

$$T_1 = a_1p_1\rho_1$$

Now let s be the maximum unit shearing stress that occurs on the extreme outer fiber. Then,

$$\frac{p_1}{s} = \frac{\rho_1}{c}$$

$$p_1 = \frac{s\rho_1}{c}$$

Then, by substitution,

$$T_1 = \frac{a_1s\rho_1^2}{c} = \frac{s}{c}a_1\rho_1^2$$

In the same way,

$$T_2 = \frac{s}{c}a_2\rho_2^2$$

$$\cdots\cdots$$

$$T_8 = \frac{s}{c}a_8\rho_8^2$$

But the total shearing torque on the circle is approximately the sum of the torques on the separate areas, or

$$T = \frac{s}{c}(a_1\rho_1^2 + a_2\rho_2^2 + a_3\rho_3^2 + \cdots + a_8\rho_8^2)$$

The true value of the torque is not obtained until the cross section of the shaft has been divided into a very large number of rings. It is shown in calculus that the summation in the parenthesis in the equation above, when the number of terms is very great, is the moment of inertia of the area of the circle about a gravity axis perpendicular to the plane of the circle. This is called the *polar moment of inertia* and will be represented by J. Then

$$T = \frac{sJ}{c} \tag{95}$$

The only kinds of shafts considered in this book are solid circular and hollow circular. Formulas will be expressed in terms of the radii of such shafts. For a circle, $J = \pi d^4/32$ where d is the diameter.

Also, $c = d/2$; then

$$T = \frac{s}{d/2}\frac{\pi d^4}{32} = \frac{\pi s d^3}{16} = 0.196sd^3 \tag{96}$$

For a hollow shaft of inside diameter d_1 and outside diameter d_2,

$$J = \frac{\pi}{32}(d_2^4 - d_1^4)$$

$$\therefore T = s\frac{\pi}{32}\frac{(d_2^4 - d_1^4)}{d_2/2} = 0.196\,\frac{s(d_2^4 - d_1^4)}{d_2} \tag{97}$$

So long as the material is not stressed beyond the elastic limit and nothing but pure shear is considered, Eqs. (96) and (97) are valid. T must be in inch-pounds and d in inches. But the torque of the internal stresses is the torque of the external forces applied to rotate or twist the shaft.

Example 1.—If $P = 400$ lb., $a = 24$ in., and the diameter of the shaft is 4 in. (Fig. 438), find the maximum shearing stress s in the outermost fiber.

Solution.—$T = Pa = 400 \times 24 = 9{,}600$ in.-lb.

Substituting in (96), we have

$$9{,}600 = 0.196 \times 64s$$

$$s = \frac{9{,}600}{0.196 \times 64} = 765 \text{ lb./sq. in.}$$

Example 2.—If 8,000 lb. per sq. in. is a safe shearing stress, what force P could be applied to the 4-in. shaft of Example 1?

Solution.—$s = 8{,}000$ lb./sq. in., $d = 4$ in., and $a = 24$ in. Then

$$24P = 8{,}000 \times 0.196 \times 64$$

$$P = \frac{8{,}000 \times 0.196 \times 64}{24} = 4{,}181 \text{ lb.}$$

Example 3.—Suppose a force $P = 3{,}000$ lb. is applied at the end of a 3-ft. lever, which is pinned first to a solid 3-in. shaft and second to a hollow shaft having an outside diameter of 5 in. but of the same area of cross section as the 3-in. shaft. Compare the maximum fiber stresses developed in each case.

Solution.—(a) $P = 3{,}000$ lb., $a = 36$ in., $d = 3$ in.

$$3{,}000 \times 36 = 0.196 s_1 \times 27$$
$$s_1 = \frac{3{,}000 \times 36}{0.196 \times 27} = 20{,}410 \text{ lb./sq. in.}$$

(b) The area of the cross section of the hollow shaft is

$$\frac{\pi 25}{4} - \frac{\pi}{4} d_1^2$$

The area of the solid shaft is $(\pi/4) \times 9$.
Then

$$\frac{\pi}{4} \times 25 - \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 9$$
$$\therefore d_1 = 4$$

Substituting in (97),

$$3{,}000 \times 36 = \frac{0.196 s_2 (625 - 256)}{5}$$
$$s_2 = \frac{3{,}000 \times 36 \times 5}{0.196 \times 369} = 7{,}500 \text{ lb./sq. in. approximately}$$

Example 4.—Find the diameter of a shaft on which a torque of 157,500 in.-lb. is acting, so that the maximum fiber stress is 8,000 lb. per sq. in.

Solution.—$T = 157{,}500$, $s = 8{,}000$.

$$157{,}500 = 0.196 \times 8{,}000 d^3$$
$$d^3 = \frac{157{,}500}{0.196 \times 8{,}000} = 100.5$$
$$d = 4.64 \text{ in.}$$

The *strength* of a shaft depends on the torque it can transmit while a safe shearing stress in the fibers is not exceeded. The strength of two shafts may then be compared by finding the ratios of their safe torques.

Example 5.—A solid shaft with a diameter of 3 in. has the same cross-sectional area as a hollow shaft of outside diameter of 5 in. Compare their strengths.

Solution.—In Example 3, the inside diameter of the hollow shaft was found to be 4 in. Then

$$T_1 = \frac{0.196 \times s(5^4 - 4^4)}{5} = 0.196 \times 73.8s$$

for the hollow shaft.

$$T_2 = 0.196s \times 3^3 = 0.196 \times 27s$$

for the solid shaft.

Dividing T_1 by T_2,

$$\frac{T_1}{T_2} = \frac{0.196 \times 73.8s}{0.196 \times 27s} = \frac{73.8}{27} = 2.73$$

That is, the hollow shaft is 2.73 times as strong as a solid shaft of equal cross-sectional area.

208. Angle of Twist.—In some types of machinery, it is desirable to know the amount of the angle of twist in a shaft, due to the torque. Let Fig. 440 represent a part of a shaft subjected to a uniform torque throughout. *AB* represents an element of the untwisted shaft, and *AB′* is the same element after torque has been applied. Owing to the torque, the radius *OB* assumes a position *OB′*. The angle *BOB′* is the angle of twist and will be called α. The displacement from *B* to *B′* is a shearing displacement. For the fiber represented by *AB* the deformation in the length *l* is *BB′*. The unit deformation

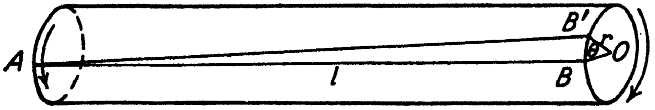


FIG. 440.

$$\delta_s = \frac{BB'}{l}, \qquad BB' = r\Theta$$

where Θ is the angle *BOB′* taken in radians.

Also, from the relations from Hooke's law,

$$\delta_s = \frac{s}{E}$$

Then

$$\frac{r\Theta}{l} = \frac{s}{E}$$

Solving, we have

$$\Theta = \frac{sl}{Er} \qquad (t)$$

It is seen that the angle of twist is given in terms of the fiber stress. In the preceding equation, s and E are in pounds per square inch and l and r are in inches.

In Eq. (95),

$$s = \frac{Tc}{J}$$

Substituting for s in Eq. (t) produces

$$\Theta = \frac{Tcl}{JEc} = \frac{Tl}{JE} \qquad (98)$$

where Θ is expressed in terms of torque. Attention is called to the fact that the angle of twist increases with the length of the shaft and decreases as the radius increases.

Example 1.—What is the length of a ½-in. steel rod that can be twisted through one half revolution without exceeding a shearing stress of 12,000 lb. per sq. in.?

Solution.—$s = 12{,}000$ lb. per sq. in., $\Theta = \pi$ radians, $c = \frac{1}{4}$ in. From Eq. (98),

$$l = \frac{Ec\Theta}{s} = \frac{12{,}000{,}000 \times \frac{1}{4}\pi}{12{,}000} = 786 \text{ in.} = 65.4 \text{ ft.}$$

Example 2.—A steel rod ¾ in. in diameter and 12 in. long is subjected to a torque of 1,600 in.-lb. What will be the angle of twist?

Solution.—$d = \frac{3}{4}$ in., $J = \pi r^4/2 = (\pi/2)(\frac{3}{8})^4 = 0.0311$, $T = 1{,}600$ in.-lb., $E = 12{,}000{,}000$.

From Eq. (98),

$$\Theta = \frac{1{,}600 \times 12}{12{,}000{,}000 \times 0.0311} = 0.0516 \text{ radians} = 2.96^\circ$$

209. Work.—When a force P that is constant in magnitude and direction acts on a body through a distance

y in the direction in which the force acts, the work done is the product of the force times the distance, or

$$W = P \times y$$

Since P is in pounds and y in feet or inches, work is in foot-pounds or inch-pounds.

For example, if $P = 20$ lb. and $y = 10$ ft.,

$$W = 20 \times 10 = 200 \text{ ft.-lb.} = 2{,}400 \text{ in.-lb.}$$

In Fig. 436, the driving force is $T_2 - T_1 = P$ on the pulley of radius r. Suppose the shaft is making N revolutions per minute (r.p.m.). In 1 revolution, the force P acts through a distance $2\pi r$. The work per revolution is $2\pi Pr$. But Pr is the moment, or torque, T. Then, in N revolutions, the total work done is $2\pi TN$ in.-lb., if r is in inches.

Now 1 *hp. is defined as* 33,000 *ft.-lb. per min., or* 33,000 × 12 *in.-lb.* Then, if horsepower is represented by H,

$$H = \frac{2\pi TN}{33{,}000 \times 12} \tag{99}$$

where T is always in inch-pounds.

From Eq. (96), for a solid shaft,

$$H = \frac{2\pi \times 0.196sd^3N}{33{,}000 \times 12} = \frac{sd^3N}{321{,}000} \tag{100}$$

From Eq. (97), for a hollow shaft,

$$H = \frac{2\pi \times 0.196s(d_2^4 - d_1^4)N}{33{,}000 \times 12d_2} = \frac{s(d_2^4 - d_1^4)N}{321{,}000d_2} \tag{101}$$

In the above formula, the student must remember that d is in inches, s a safe working shearing fiber stress, and N the number of revolutions per minute.

Example 1.—Find the horsepower transmitted by a 3-in. steel shaft, making 200 r.p.m., with a fiber stress of 7,000 lb. per sq. in.

Solution.—$s = 7{,}000$, $d = 3$, and $N = 200$.

$$H = \frac{7{,}000 \times 27 \times 200}{321{,}000} = 117.8$$

Example 2.—Find the diameter of a solid steel shaft to transmit 150 hp. at 225 r.p.m., a safe working stress of 6,000 lb. per sq. in. being used.

Solution.—From Eq. (100),

$$H = \frac{sd^3N}{321{,}000}$$

But

$$H = 150, \qquad N = 225, \qquad \text{and } s = 6{,}000$$

$$150 = \frac{6{,}000d^3225}{321{,}000}$$

$$d^3 = 35.7$$

$$\therefore d = \sqrt[3]{35.7} = 3.3 \text{ in.}$$

Example 3.—A solid shaft 8 in. in diameter has the same cross-sectional area as a hollow shaft of the same material, with inside diameter 6 in. Compare the horsepower that each will deliver when making the same number of r.p.m.

Solution.—Let d_2 be the outside diameter of the hollow shaft. Its area is

$$\frac{\pi d_2^2}{4} - \frac{\pi 36}{4} = \frac{\pi}{4}(d_2^2 - 36)$$

The area of the solid shaft is $\frac{\pi}{4} \times 64$.

$$\frac{\pi}{4}(d_2^2 - 36) = \frac{\pi}{4} \times 64$$

$$\therefore d_2 = 10 \text{ in.}$$

From Eq. (100),

$$H = \frac{8^3sN}{321{,}000} = \frac{512sN}{321{,}000}$$

From Eq. (101),

$$H = \frac{(10^4 - 6^4)Ns}{321{,}000 \times 10} = \frac{870.4sN}{321{,}000}$$

$$\therefore \frac{H \text{ (hollow)}}{H \text{ (solid)}} = \frac{870.4}{512} = 1.7$$

That is, the hollow shaft will transmit 1.7 times as many horsepower as the solid shaft, of the same area, material, and under the same fiber stress.

210. Tension Due to Shear.—In Fig. 440*a*, rectangle *A* shows the direct and also the induced shearing stresses due to torsion. There is a plane 1 – 1 at any particular point on the surface of the shaft at which a tensile stress is produced as a result of the shearing stresses. It is due to this tension that the failure of a shaft is often found to take place along a plane corresponding to plane 1 – 1.

Example 1.—In Example 2, Art. 207, what is the unit tensile stress in the surface of the shaft?

Solution.

$$s_s = 8{,}000 \text{ lb./sq. in.}$$
$$s_t = s_s \sin (2\Theta) \text{ from (51)}$$
$$s_t = 8{,}000 \sin (2 \times 45°) = 8{,}000 \text{ lb./sq. in.}$$

Thus, due to the original shearing stresses resulting from the torque, a tensile stress of the same value is produced on a plane at an angle of 45° with the direction of the shears. There is an equal compressive stress on a plane at right angles to the plane of tension.

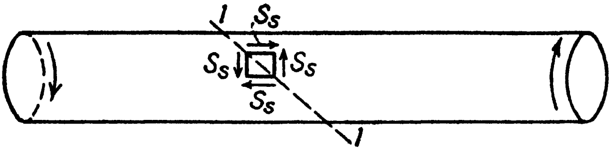


FIG. 440*a*.

211. Combined Torsion and Bending.—In the previous articles on torsion, the assumption was made that the shaft resisted shear alone. But in many cases, as of a shaft driving a machine by means of a belt over a pulley and also the shaft for a flywheel, the shaft is subjected to bending as well as twisting. A shaft cannot carry so high a stress in torsion because of the bending, and conversely the bending stress must be lowered because of the torsion. The mathematics and mechanics involved in a discussion of this problem are beyond the scope of this book.

Formulas for equivalent twisting and bending moments have been arrived at, and will be given here. By *equivalent twisting moment* is meant a twisting moment large enough to care for both bending and twisting, in a shaft designed to carry such a twisting moment. The equivalent twisting moment is represented by T_e, and the equivalent bending moment by M_e. The formulas are

$$M_e = \frac{M}{2} + \frac{1}{2}\sqrt{M^2 + T^2} \tag{102}$$

and

$$T_e = \sqrt{M^2 + T^2} \tag{103}$$

where M stands for bending moment and T for twisting moment, as before.

For round shafts,

$$M_e = \frac{pI}{c} = \frac{p\pi d^4}{64 \times (d/2)} = \frac{\pi p d^3}{32} = \frac{pd^3}{10.2} \tag{104}$$

and

$$T_e = \frac{sJ}{c} = \frac{s\pi d^4}{32 \times (d/2)} = \frac{\pi s d^3}{16} = \frac{sd^3}{5.1} \tag{105}$$

Equations (104) and (105) are expressed in terms of the diameter and will be found very useful. In case there is both twisting and bending, both formulas should be figured for the diameter, and the largest value for d is the one to use.

Example 1.—A wrought-iron shaft 3 in. in diameter, making 150 r.p.m., is supported in bearings 6 ft. apart. If the load of 250 lb. is brought to the middle of the shaft by a belt and a pulley, what horsepower H can be transmitted by the shaft if the maximum shearing stress is 6,000 lb. per sq. in.?

Solution.—By Eq. (105), the maximum equivalent torque is

$$T_e = \frac{6{,}000 \times 27}{5.1} = 31{,}760 \text{ in.-lb.}$$

Also, from Table VIII,

$$M = \frac{1}{4}Wl = \frac{250 \times 6 \times 12}{4} = 4{,}500 \text{ in.-lb.}$$

But $T_e = \sqrt{M^2 + T^2}$.

Then

$$31{,}760 = \sqrt{(4{,}500)^2 + T^2}$$

Squaring both sides of the equation and solving for T,

$$T = \sqrt{31{,}760^2 - 4{,}500^2} = 31{,}400 \text{ in.-lb. approximately}$$

Now,

$$H = \frac{2\pi TN}{12 \times 33{,}000} = \frac{2\pi \times 31{,}400 \times 150}{12 \times 33{,}000} = 74.8$$

Example 2.—Find the diameter of a shaft subjected to a bending moment of 6,000 in.-lb. and a twisting moment of 8,000 in.-lb., if the bending stress does not exceed 10,000 lb. per sq. in. and the twisting stress does not exceed 8,000 lb. per sq. in.

Solution.—$M = 6{,}000$, and $T = 8{,}000$.

By Eqs. (102) and (103),

$$M_e = \frac{6{,}000}{2} + \frac{1}{2}\sqrt{(6{,}000)^2 + (8{,}000)^2} = 3{,}000 + 5{,}000 = 8{,}000 \text{ in.-lb.}$$

$$T_e = \sqrt{(6{,}000)^2 + (8{,}000)^2} = 10{,}000 \text{ in.-lb.}$$

In the formulas for Me and Te in terms of s and d, the s stands for allowable bending stress in (104), and shearing stress in (105). Then

$$8{,}000 = \frac{10{,}000d^3}{10.2}, \qquad d = \sqrt[3]{8.16} = 2.01$$

$$10{,}000 = \frac{8{,}000d^3}{5.1}, \qquad d = \sqrt[3]{6.375} = 1.85$$

Then the diameter necessary is 2.01.

Example 3.—Find the diameter of a steel shaft in bearings 10 ft. apart to transmit 100 hp. at 300 r.p.m. There is a load of 500 lb. at the middle, and the maximum allowable stress in bending is 10,000 lb. per sq. in. and 7,000 in twisting.

Solution.—Now $M = \frac{1}{4}Wl = \frac{1}{4} \times 500 \times 120 = 15{,}000$ in.-lb.

To find T, use Eq. (99). Then

$$100 = \frac{2\pi T300}{12 \times 33{,}000}$$

$$T = 21{,}000 \text{ in.-lb.}$$

Then, by Eqs. (102) and (103),

$$M_e = \frac{15{,}000}{2} + \frac{1}{2}\sqrt{(15{,}000)^2 + (21{,}000)^2} = 7{,}500 + 13{,}000 = 20{,}500 \text{ in.-lb.}$$

$$T_e = \sqrt{(15{,}000)^2 + (21{,}000)^2} = 26{,}000 \text{ in.-lb.}$$

From Eqs. (104) and (105),

$$20{,}500 = \frac{10{,}000d^3}{10.2}$$

$$d = \sqrt[3]{20.91} = 2.8 \text{ in.}$$

$$26{,}000 = \frac{7{,}000d^3}{5.1}$$

$$d = \sqrt[3]{18.94} = 2.7 \text{ in.}$$

Then a 2.8-in. shaft is necessary.

212. Shaft Couplings.—It is frequently necessary to connect two pieces of shafting end to end so that they will act as a single shaft. Such a connection is known as a "flange coupling." In one type the flanges are forged as a part of the shaft, and the flanges at the ends of successive lengths of shafts are bolted together. In the more common type a flange and hub are either cast or forged and then bored to fit the shaft. The hubs of the coupling are then keyed to the ends of successive shafts, and the couplings are bolted together.

Problems

389. What will be the r.p.m. of a 12-in. steel shaft to transmit 6,000 hp.? $s_s = 9{,}000$ lb./sq. in.

390. If $s_s = 10{,}000$ lb./sq. in., what should be the diameter of a solid steel shaft to transmit 180 hp. at 150 r.p.m.?

391. What horsepower can be transmitted by a 3-in. solid steel shaft rotating at 275 r.p.m.? $s_s = 9{,}000$.

392. If $s_s = 2{,}000$ lb./sq. in., what horsepower can be transmitted by a solid cast-iron shaft 5 in. in diameter rotating at 100 r.p.m.?

393. A wrought-iron shaft is to transmit 140 hp. at 90 r.p.m. What should be the diameter if $s = 8{,}000$ lb./sq. in.?

394. A hollow cast-iron shaft 8 in. outside diameter and 6 in. inside diameter is to transmit 500 hp. At what r.p.m. should it be run? $s_s = 2{,}500$ lb./sq. in.

395. What is the ratio of the horsepower developed, respectively, by 2-in., 4-in., and 6-in. shafts rotating at 90 r.p.m.? $s_s = 9{,}000$ lb./sq. in.

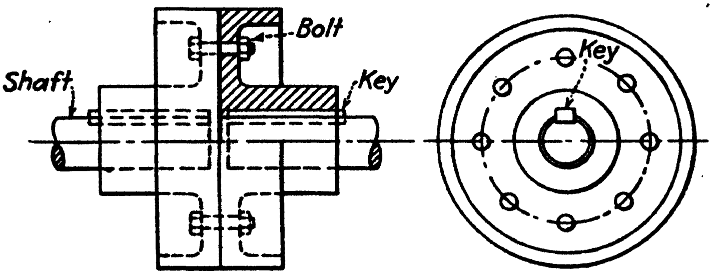
396. A 4-in. shaft is transmitting 30 hp. at 180 r.p.m. and is subjected to a bending moment of 8,400 in.-lb. What will be the maximum tensile and shearing stresses developed?

397. If the bending moment is equal to the twisting moment, what should be the diameter of a solid steel shaft that is to transmit 120 hp. at 125 r.p.m.?

398. A steel shaft 3 in. in diameter rotating at 160 r.p.m. is supported in bearings 14 ft. apart. Owing to the weight of the shaft and the pulley and the downward pull of the belt, a load of 250 lb. may be considered to act at the center. If the shearing stress is 8,000 lb. per sq. in., what horsepower can be transmitted?

399. What should be the diameter of a steel shaft in fixed bearings 9 ft. apart, if it is to transmit 90 hp. at 240 r.p.m.? The bending effect is equivalent to a moment produced by a load of 600 lb. placed midway between the bearings.

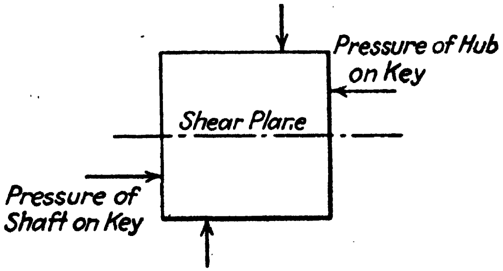
400. Find the size of a hollow steel shaft with the inside diameter equal to one-half of the outside diameter. The moment due to the weight of the shaft and pulley and the belt pulls amounts to 12,000 in.-lb. The twisting moment is equal to 15,000 in.-lb. $s_s = 8{,}000$ lb./sq. in.



Flange Coupling

Fig. 441.

401. Figure 441 shows a coupling that transmits 24 hp. while rotating at 130 r.p.m. The driving pulley is fastened to a 3-in. shaft by a key ¾ in. wide, ½ in. thick, and 5 in. long. What are the shearing and bearing stresses in the key (Fig. 441*a*).



Forces Acting on Key

Fig. 441*a*.

402. If the shearing and bearing stresses are limited, respectively, to 8,000 and 16,000 lb. per sq. in., how much horsepower can be transmitted by the coupling of Fig. 441 if the shaft is rotating at 180 r.p.m.?

403. With allowable stresses as in Prob. 409, what safe torque can a pulley transmit to a 9-in. shaft if the pulley is keyed to the shaft by means of two keys, each 1 by 1 in. and 10 in. long (Fig. 441*a*)? What horsepower would correspond to a speed of 150 r.p.m.?

404. A torque of 90,000 in.-lb. is transmitted from a 6-in. shaft to a pulley by means of two keys each 6 in. long. If $s_s = 10,000$ and $s_c = 20,000$ lb./sq. in., respectively, what should be the cross section of the keys, good contact being assumed?

405. What torque can be transmitted by the eight bolts shown in the coupling of Fig. 441? The bolts ⅞ in. in diameter are in bearing against the ¾-in. web of the pulley and are set in a circle with a radius of 5.5 in. Allowable stresses are as in Prob. 404.

406. The shearing stress in a solid steel shaft 9 in. in diameter is 9,000 lb. per sq. in. What horsepower is it transmitting at 150 r.p.m.? The flange couplings have 12 bolts 1½ in. in diameter arranged in a circle with 14-in. radius. The web of coupling is 1-in. thick. What are the stresses in the bolts?

407. A steel shaft 1 in. in diameter is twisted through an angle of 2° in a length of 10 ft. What will be the unit shearing stress?

408. A steel shaft 10 ft. long is subjected to a torque of 1,800 in.-lb., an angle of twist of 2° being thus caused. If the shearing stress is not to exceed 12,000 lb. per sq. in., what diameter is needed?

409. A steel shaft 5 in. in diameter transmits 180 hp. at a speed of 240 r.p.m. The length of shaft subjected to the torque is 10 ft. What is the angle of twist? What is the unit shearing stress?

410. What are the tensile stresses resulting from the shear in Probs. 391, 400 and 409?

APPENDIX

TABLE IX.—TRIGONOMETRY FORMULAS AND DEFINITIONS

$$\sin A = \frac{a}{c} = \frac{\text{op. side}}{\text{hyp.}}$$

$$\cos A = \frac{b}{c} = \frac{\text{adj. side}}{\text{hyp.}}$$

$$\tan A = \frac{a}{b} = \frac{\text{op. side}}{\text{adj. side}}$$

$$\cot A = \frac{b}{a} = \frac{\text{adj. side}}{\text{op. side}}$$

$$\sin B = \frac{b}{c} = \frac{\text{op. side}}{\text{hyp.}}$$

$$\cos B = \frac{a}{c} = \frac{\text{adj. side}}{\text{hyp.}}$$

$$\tan B = \frac{b}{a} = \frac{\text{op. side}}{\text{adj. side}}$$

$$\cot B = \frac{a}{b} = \frac{\text{adj. side}}{\text{op. side}}$$

$$\sin (90 - A) = \cos A$$

$$\cos (90 - A) = \sin A$$

$$\sin (90 - B) = \cos B$$

$$\cos (90 - B) = \sin B$$

$$\sin (90 + \Theta) = \cos \Theta$$

$$\cos (90 + \Theta) = - \sin \Theta$$

$$\sin (180 - \Theta) = \sin \Theta$$

$$\cos (180 - \Theta) = - \cos \Theta$$

$$\sin^2 \Theta + \cos^2 \Theta = 1$$

$$\sec^2 \Theta = 1 + \tan^2 \Theta$$

$$\csc^2 \Theta = 1 + \cot^2 \Theta$$

$$\tan \Theta = \frac{\sin \Theta}{\cos \Theta}$$

$$\cot \Theta = \frac{\cos \Theta}{\sin \Theta}$$

$$\sin (- \Theta) = - \sin \Theta$$

$$\cos (- \Theta) = \cos \Theta$$

TABLE X.—WIDE-FLANGE BEAMS

Depth	Weight per foot	Area of section	Flange		Web thickness	I_{1-1}	S	r	I_{2-2}	S	r
			Width	Thickness							
36	240	70.6	16.5	1.32	0.79	1,572.4	873.6	14.92	920.1	111.5	3.61
36	160	47.09	12.0	1.02	0.653	9,738.8	541.0	14.38	275.4	45.9	2.42
33	220	64.73	15.75	1.275	0.775	12,312.1	740.6	13.79	782.4	99.0	3.48
	200	58.79	15.75	1.15	0.715	11,048.2	669.6	13.71	691.7	87.8	3.43
33	152	44.71	11.5	1.005	0.635	8,147.6	486.4	13.5	256.1	44.3	2.39
	125	36.78	11.5	0.805	0.57	6,354.7	385.1	13.14	188.2	32.7	2.26
30	210	61.78	15	1.315	0.775	9,872.4	649.9	12.64	707.9	93.7	3.38
	180	52.89	15	1.125	0.67	8,328.2	555.2	12.55	585.6	78.1	3.33
30	132	38.83	10.5	1.0	0.615	5,753.1	379.7	12.17	185.0	35.1	2.18
	116	34.13	10.5	0.85	0.564	4,919.1	327.9	12.0	153.2	29.2	2.12
27	177	52.1	14.0	1.19	0.725	6,728.6	492.8	11.36	518 9	73.7	3.16
	154	45.3	14.0	1.035	0.635	5,775.8	472.8	11.29	437.6	62.5	3.11
	114	33.53	10.0	0.932	0.57	4,080.5	299.2	11.03	149.6	29.7	2.11
	98	28.82	10.0	0.792	0.50	3,446.5	255.3	10.94	122.9	24.6	2.07
24	150	44.1	14.0	1.055	0.628	4,733.5	385.5	10.36	452.5	64.3	3.20
	130	38.21	14.0	0.90	0.565	4,009.5	330.7	10.24	375.2	53.6	3.13
	120	35.29	12.0	0.93	0.556	3,635.3	299.1	10.15	254.0	42.0	2.68
	100	29.43	12.0	0.775	0.468	2,987.3	248.9	10.08	203.5	33.9	2.63
	94	27.63	9.0	0.872	0.516	2,683.0	220.9	9.85	102.2	22.6	1.92
	80	23.54	9.0	0.727	0.455	2,229.7	185.8	9.73	82.4	18.3	1.87
21	132	38.81	13.0	1.02	0.614	3,141.6	294.8	9.0	353.8	54.1	3.02
	112	32.93	13.0	0.865	0.527	2,620.6	249.6	8.92	289.7	44.6	2.96
	89	26.15	9.0	0.865	0.537	1,919.2	182.8	8.57	99.4	22.1	1.95
	59	17.36	8.25	0.575	0.39	1,246.8	119.3	8.47	49.2	12.0	1.68
18	114	36.5	11.75	0.991	0.595	2,033.8	220.1	7.79	255.6	43.2	2.76
	96	28.22	11.75	0.831	0.512	1,674.7	184.4	7.70	206.8	35.2	2.71
	70	20.56	8.75	0.751	0.438	1,153.9	128.2	7.49	78.5	17.9	1.95
	50	14.71	7.5	0.57	0.358	800.6	89.0	7.38	37.2	9.9	1.59
16	88	25.87	11.5	0.795	0.504	1,222.6	151.3	6.87	185.2	32.2	2.67
	58	17.04	8.5	0.645	0.407	746.4	94.1	6.62	60.5	14.3	1.88
	40	11.77	7.0	0.503	0.307	515.5	64.4	6.62	26.5	7.6	1.5
14 × 16	426.0	125.25	16.695	3.003	1.875	6,610.3	707.4	7.26	2,359.5	282.7	4.32
	356	104.68	16.42	2.563	1.6	5,179.4	583.6	7.03	1,895.7	230.9	4.26
	300	88.2	16.175	2.188	1.355	4,149.5	488.2	6.86	1,546.0	191.2	4.19
	255	74.98	15.99	1.873	1.17	3,372.6	412.0	6.71	1,278.1	159.9	4.13
	202	59.39	15.75	1.503	0.93	2,538.8	324.9	6.54	979.7	124.4	4.06
	150	44.08	15.515	1.128	0.695	1,786.9	240.2	6.37	702.5	90.6	3.99

TABLE X—*continued*

Depth	Weight per foot	Area of section	Flange		Web thickness	I_{1-1}	S	r	I_{2-2}	S	r
			Width	Thickness							
14 × 14½	127	37.33	14.69	0.998	0.61	1,476.7	202.0	6.29	527.6	71.8	3.76
	103	30.26	14.575	0.813	0.495	1,165.8	163.6	6.21	419.7	57.6	3.72
	87	25.56	14.5	0.688	0.42	966.9	138.1	6.15	349.7	48.2	3.7
	78	22.94	12.0	0.718	0.428	851.2	121.1	6.09	206.9	34.5	3.00
	61	17.94	10.0	0.643	0.378	641.5	92.2	5.98	107.3	21.5	2.45
14	43	12.65	8.0	0.528	0.308	429.0	62.7	5.82	45.1	11.3	1.89
	30	8.81	6.733	0.383	0.27	289.6	41.8	5.73	17.5	5.2	1.41
12	161	47.38	12.515	1.486	0.905	1,541.8	222.2	5.70	486.2	77.7	3.2
	120	35.31	12.32	1.106	0.71	1,071.7	163.4	5.51	345.1	56.0	3.13
	99	29.09	12.19	0.921	0.58	858.5	134.7	4.43	278.2	45.7	3.09
	65	19.11	12.0	0.606	0.39	533.4	88.0	5.28	174.6	29.1	3.02
	53	15.59	10.0	0.576	0.345	426.2	70.7	5.23	96.1	19.2	2.48
	40	11.77	8.0	0.516	0.294	310.1	51.9	5.13	44.1	11.0	1.94
	25	7.39	6.5	0.355	0.24	183.4	30.9	4.98	14.5	4.5	1.4
10	124	36.46	10.505	1.368	0.845	813.1	139.9	4.72	264.8	50.4	2.69
	100	29.43	10.345	1.118	0.685	625.0	112.4	4.61	206.6	39.9	2.65
	72	21.18	10.17	0.808	0.510	420.7	80.1	4.46	141.8	27.9	2.59
	49	14.4	10.0	0.558	0.34	272.9	54.6	4.35	93.0	18.6	2.54
	41	12.06	8.0	0.558	0.328	222.4	44.5	4.29	47.7	11.9	1.99
	23	6.77	5.75	0.39	0.24	120.6	24.1	4.22	11.3	3.9	1.29
8	40	11.76	8.077	0.558	0.365	146.3	35.5	3.53	49.0	12.1	2.04
	31	9.12	8.0	0.433	0.288	109.7	27.4	3.47	37.0	9.2	2.01
	24	7.06	6.5	0.398	0.245	82.5	20.8	3.42	18.2	5.6	1.61

TABLE XI.—PROPERTIES OF STANDARD I-BEAMS

Depth of beam, *inches*	Weight per foot, *pounds*	Area of section, *square inches*	Thickness of web, *inches*	Width of flange, *inches*	Moment of inertia axis 1—1, *inches*^{4}	Section modulus axis 1—1, *inches*^{3}	Radius of gyration axis 1—1, *inches*	Moment of inertia axis 2—2, *inches*^{4}	Radius of gyration axis 2—2, *inches*
3	5.7	1.64	0.17	2.33	2.5	1.7	1.23	0.46	0.53
3	6.5	1.88	0.251	2.411	2.7	1.8	1.19	0.51	0.52
3	7.5	2.17	0.349	2.509	2.9	1.9	1.15	0.59	0.52
4	7.7	2.21	0.19	2.66	6.0	3.0	1.64	0.77	0.59
4	8.5	2.46	0.253	2.723	6.3	3.2	1.60	0.83	0.58
4	9.5	2.76	0.326	2.796	6.7	3.3	1.56	0.91	0.58
4	10.5	3.05	0.40	2.87	7.1	3.5	1.52	1.00	0.57
5	10.0	2.87	0.21	3.0	12.1	4.8	2.05	1.2	0.65
5	12.25	3.56	0.347	3.137	13.5	5.4	1.95	1.4	0.63
5	14.75	4.29	0.494	3.284	15.0	6.0	1.87	1.7	0.63
6	12.5	3.61	0.23	3.33	21.8	7.3	2.46	1.8	0.72
6	14.75	4.29	0.343	3.443	23.8	7.9	2.36	2.1	0.69
6	17.25	5.02	0.465	3.565	26.0	8.7	2.28	2.3	0.68
7	15.3	4.43	0.25	3.66	36.2	10.4	2.86	2.7	0.78
7	17.5	5.09	0.345	3.755	38.9	11.1	2.77	2.9	0.76
7	20.0	5.83	0.45	3.86	41.9	12.0	2.68	3.1	0.74
8	17.5	5.13	0.22	4.0	58.4	14.6	3.38	6.2	1.10
8	20.5	5.97	0.349	4.079	60.2	15.1	3.18	4.0	0.82
8	23.0	6.71	0.441	4.171	64.2	16.0	3.09	4.4	0.81
8	25.5	7.43	0.532	4.262	68.1	17.0	3.03	4.7	0.80
9	21.8	6.32	0.29	4.33	84.9	18.9	3.67	5.2	0.90
9	25.0	7.28	0.397	4.437	91.4	20.3	3.54	5.6	0.88
9	30.0	8.76	0.561	4.601	101.4	22.5	3.40	6.4	0.85
9	35.0	10.22	0.724	4.764	111.3	24.7	3.30	7.3	0.84
10	25.4	7.38	0.31	4.66	122.1	24.4	4.07	6.9	0.97
10	30.0	8.75	0.447	4.797	133.5	26.7	3.91	7.6	0.93
10	35.0	10.22	0.594	4.944	145.8	29.2	3.78	8.5	0.91
10	40.0	11.69	0.741	5.091	158.0	31.6	3.68	9.4	0.90
12	31.8	9.26	0.35	5.00	215.8	36.0	4.83	9.5	1.01
12	35.0	10.20	0.428	5.078	227.0	37.8	4.72	10.0	0.99
12	40.8	11.84	0.46	5.25	268.9	44.8	4.77	13.8	1.08
15	42.9	12.49	0.41	5.50	441.8	58.9	5.95	14.6	1.08
15	45.0	13.12	0.452	5.542	453.6	60.5	5.88	15.0	1.07
15	50.0	14.59	0.55	5.64	481.1	64.2	5.74	16.0	1.05
15	55.0	16.06	0.648	5.738	508.7	67.8	5.68	17.0	1.03
15	60.8	17.68	0.59	6.00	609.0	81.2	5.87	26.0	1.21
18	54.7	15.94	0.46	6.00	795.5	88.4	7.07	21.2	1.15
18	60.0	17.50	0.547	6.087	837.8	93.1	6.92	22.3	1.13
18	65.0	18.98	0.629	6.169	877.7	97.5	6.80	23.4	1.11
18	70.0	20.46	0.711	6.251	917.5	101.9	6.70	24.5	1.09
20	65.4	19.08	0.50	6.25	1169.5	116.9	7.83	27.9	1.21
20	70.0	20.42	0.567	6.317	1214.2	121.4	7.71	28.9	1.19
20	75.0	21.90	0.641	6.391	1263.5	126.3	7.60	30.1	1.17
24	81.4	23.74	0.60	7.00	1466.3	146.6	7.86	45.8	1.39
24	85.0	24.80	0.653	7.053	1501.7	150.2	7.78	47.0	1.38
24	90.0	26.26	0.726	7.126	1550.3	155.0	7.68	48.7	1.36
24	95.0	27.74	0.800	7.20	1599.7	160.0	7.59	50.5	1.35
24	100.0	29.20	0.873	7.273	1648.3	164.8	7.51	52.4	1.34

Table XII.—Properties of Standard Channels

Depth of channel, *inches*	Weight per foot, *pounds*	Area of section, *square inches*	Thickness of web *inches*	Width of flange, *inches*	Axis 1–1			Axis 2–2			
					I, *inches*^{4}	S, *inches*^{3}	r, *inches*	I, *inches*	S, *inches*^{3}	r, *inches*	x, *inches*
3	4.1	1.19	0.17	1.41	1.6	1.1	1.17	0.20	0.21	0.41	0.44
3	5.00	1.46	0.26	1.50	1.8	1.2	1.12	0.25	0.24	0.41	0.44
3	6.00	1.75	0.36	1.60	2.1	1.4	1.08	0.31	0.27	0.42	0.46
4	5.4	1.55	0.18	1.58	3.8	1.9	1.56	0.32	0.29	0.45	0.46
4	6.25	1.82	0.25	1.65	4.1	2.1	1.50	0.38	0.32	0.45	0.46
4	7.25	2.13	0.33	1.73	4.6	2.3	1.46	0.44	0.35	0.46	0.46
5	6.50	1.95	0.19	1.75	7.4	3.0	1.95	0.48	0.38	0.50	0.49
5	9.00	2.63	0.33	1.89	8.9	3.5	1.83	0.64	0.45	0.49	0.48
5	11.50	3.36	0.48	2.04	10.4	4.2	1 75	0.82	0.54	0.49	0.51
6	8.2	2.38	0.20	1.92	13.0	4.3	2.34	0.70	0.50	0.54	0.52
6	10.50	3.09	0.32	2.04	15.1	5.0	2.21	0.88	0.57	0.53	0.50
6	13.00	3.82	0.44	2.16	17.3	5.8	2.13	1.07	0.65	0.53	0.52
6	15.50	4.56	0.56	2.28	19.5	6.5	2.07	1.28	0.74	0.53	0.55
7	9.8	2.85	0.21	2.09	21.1	6.0	2.72	0.98	0.63	0.59	0.55
7	12.25	3.58	0.32	2.20	24.2	6.9	2.59	1.19	0.71	0.57	0.53
7	14.75	4.32	0.42	2.30	27.2	7.8	2.50	1.40	0.79	0.57	0.53
7	17.25	5.05	0.53	2.41	30.2	8.6	2.44	1.62	0.87	0.56	0.55
7	19.75	5.79	0.63	2.51	33.2	9.5	2.39	1.85	0.96	0.56	0.58
8	11.5	3.35	0.22	2.26	32.3	8.1	3.10	1.33	0.79	0.63	0.58
8	13.75	4.04	0.31	2.35	36.0	9.0	2.98	1.55	0.87	0.62	0.56
8	16.25	4.78	0.40	2.44	39.9	10.0	2.89	1.78	0.95	0.61	0.56
8	18.75	5.51	0.49	2.53	43.8	11.0	2.82	2.01	1.02	0.60	0.57
8	21.25	6.25	0.58	2.62	47.6	11.9	2.76	2.2	1.11	0.60	0.59
9	13.4	3.89	0.23	2.43	47.3	10.5	3.49	1.77	.97	0.67	0.61
9	15 00	4.41	0.29	2.49	50.9	11.3	3.40	1.95	1.03	0.66	0.59
9	20.00	5.88	0.45	2.65	60.8	13.5	3.21	2.45	1.19	0.65	0.59
9	25.00	7.35	0.61	2.81	70.7	15.7	3.10	2.98	1.36	0.64	0.62
10	15.3	4.46	0.24	2.60	66.9	13.4	3.87	2.30	1.17	0.72	0.64
10	20.00	5.88	0.38	2.74	78.7	15.7	3.66	2.85	1.34	0.70	0.61
10	25.00	7.35	0.53	2.89	91.0	18.2	3.52	3.40	1.50	0.68	0.62
10	30.00	8.82	0.68	3.04	103.2	20.6	3.42	3.99	1.67	0.67	0.65
10	35.00	10.29	0.82	3.18	115.5	23.1	3.35	4.66	1.87	0.67	0.69
12	20.7	6.03	0.28	2.94	128.1	21.4	4.61	3.91	1.75	0.81	0.70
12	25.00	7.35	0.39	3.05	144.0	24.0	4.43	4.53	1.91	0.78	0.68
12	30.00	8.82	0.51	3.17	161.6	26.9	4.28	5.21	2.09	0.77	0.68
12	35.00	10.29	0.64	3.30	179.3	29.9	4.17	5.90	2.27	0.76	0.69
12	40.00	11.76	0.76	3.42	196.9	32.8	4.09	6.63	2.46	0.75	0.72
15	33.9	9.90	0.40	3.40	312.6	41.7	5.62	8.2	3.2	0.91	0.79
15	35.00	10.23	0.43	3.43	318.7	42.7	5.57	8.4	3.2	0.91	0.79
15	40.00	11.70	0.52	3.52	346.3	46.3	5.44	9.3	3.4	0.89	0.78
15	45.00	13.17	0.62	3.62	373.9	50.0	5.32	10.3	3.6	0.88	0.79
15	50.00	14.64	0.72	3.72	401.4	53.7	5.23	11.2	3.8	0.87	0.80
15	55.00	16.11	0.82	3.82	420.0	57.4	5.16	12.1	4.1	0.87	0.82

TABLE XIIIA.—PROPERTIES OF STANDARD EQUAL LEG ANGLES

Size	Area of Section A	Axis 1–1				Axis 3–3
		I	r	S	$\bar{x}$	Min r
Inches	*Inches*^{2}	*Inches*^{4}	*Inches*	*Inches*^{3}	*Inches*	*Inches*
8x8x1	15.00	89.0	2.44	15.8	2.37	1.56
8x8x¾	11.44	69.7	2.47	12.2	2.28	1.57
8x8x½	7.75	48.6	2.51	8.4	2.19	1.58
6x6x1	11.00	35.5	1.80	8.6	1.86	1.16
6x6x¾	8.44	28.2	1.83	6.7	1.78	1.17
6x6x½	5.75	19.9	1.86	4.6	1.68	1.18
5x5x1	9.00	19.6	1.48	5.8	1.61	0.96
5x5x¾	6.94	15.7	1.50	4.5	1.52	0.97
5x5x½	4.75	11.3	1.54	3.2	1.43	0.98
4x4x¾	5.44	7.7	1.19	2.8	1.27	0.77
4x4x½	3.75	5.6	1.22	2.0	1.18	0.78
4x4x⅜	2.86	4.4	1.23	1.5	1.14	0.79
3½x3½x¾	4.69	5.0	1.03	2.1	1.15	0.67
3½x3½x½	3.25	3.6	1.06	1.5	1.06	0.68
3½x3½x¼	1.69	2.0	1.09	0.79	0.97	0.69
3x3x⅝	3.36	2.6	0.88	1.3	0.98	0.57
3x3x½	2.75	2.2	0.90	1.1	0.93	0.58
3x3x⅜	2.11	1.8	0.91	0.83	0.89	0.58
2x2x⅜	1.36	0.48	0.59	0.35	0.64	0.39
2x2x¼	0.94	0.35	0.61	0.25	0.59	0.39
2x2x⅛	0.48	0.19	0.63	0.13	0.55	0.40

TABLE XIIIB.—UNEQUAL LEG ANGLES

		Area of section A	I_{1-1}	S	r	x	I_{2-2}	S	r	y	r min.
	1	13.00	80.8	15.1	2.49	2.65	38.8	8.9	1.73	1.65	1.28
8 × 6	⅞	11.48	72.3	13.4	2.51	2.61	34.9	7.9	1.74	1.61	1.28
	¾	9.94	63.4	11.7	2.53	2.56	30.7	6.9	1.76	1.56	1.29
	⅝	8.36	54.1	9.9	2.54	2.52	26.3	5.9	1.77	1.52	1.30
	½	6.75	44.3	8.0	2.56	2.47	21.7	4.8	1.79	1.47	1.30
	1	11.00	69.6	14.1	2.52	30.5	11.6	3.9	1.03	1.05	0.85
	⅞	9.73	62.4	12.5	2.53	3.00	10.5	3.5	1.04	1.00	0.85
	¾	8.44	54.9	10.9	2.55	2.95	9.4	3.1	1.05	0.95	0.85
8 × 4	⅝	7.11	46.9	9.2	2.56	2.91	8.1	2.6	1.07	0.91	0.86
	½	5.75	38.5	7.5	2.59	2.86	6.7	2.2	1.08	0.86	0.86
	1	10.00	47.7	10.8	2.18	2.60	11.2	3.9	1.06	1.10	0.85
	⅞	8.86	42.9	9.7	2.20	2.55	10.2	3.5	1.07	1.05	0.86
	¾	7.69	37.8	8.4	2.22	2.51	9.1	3.0	1.09	1.01	0.86
7 × 4	⅝	6.49	32.4	7.1	2.24	2.46	7.8	2.6	1.10	0.96	0.86
	½	5.25	26.7	5.8	2.25	2.42	6.5	2.1	1.11	0.92	0.87
	⅜	3.99	20.6	4.4	2.27	2.37	5.1	1.6	1.13	0.87	0.88
	1	9.00	30.8	8.0	1.85	2.17	10.8	3.8	1.09	1.17	0.85
	⅞	7.93	27.7	7.2	1.86	2.12	9.8	3.4	1.11	1.12	0.86
	¾	6.94	24.5	6.2	1.88	2.08	8.7	3.0	1.12	1.08	0.86
6 × 4	⅝	5.86	21.1	5.3	1.90	2.03	7.5	2.5	1.13	1.03	0.86
	½	4.75	17.4	4.3	1.91	1.99	6.3	2.1	1.15	0.99	0.87
	⅜	3.61	13.5	3.3	1.93	1.94	4.9	1.6	1.17	0.94	0.88
	¾	5.81	13.9	4.3	1.55	1.75	5.6	2.2	0.98	1.00	0.75
	⅝	4.92	12.0	3.7	1.56	1.70	4.8	1.9	0.99	0.95	0.75
5 × 3½	½	4.00	10.0	3.0	1.58	1.66	4.0	1.6	1.01	0.91	0.75
	⅜	3.05	7.8	2.3	1.60	1.61	3.2	1.2	1.02	0.86	0.76
	¾	5.06	7.3	2.8	1.20	1.34	5.2	2.1	1.01	1.09	0.72
	⅝	4.30	6.4	2.4	1.22	1.29	4.5	1.8	1.03	1.04	0.72
4 × 3½	½	3.50	5.3	1.9	1.23	1.25	3.8	1.5	1.04	1.00	0.72
	⅜	2.67	4.2	1.5	1.25	1.21	3.0	1.2	1.06	0.96	0.73
	¾	4.69	6.9	2.7	1.22	1.42	3.3	1.6	0.84	0.92	0.64
	⅝	3.98	6.0	2.3	1.23	1.37	2.9	1.4	0.85	0.87	0.64
4 × 3	½	3.25	5.0	1.9	1.25	1.33	2.4	1.1	0.86	0.83	0.64
	⅜	2.48	4.0	1.5	1.26	1.28	1.9	0.87	0.88	0.78	0.64
	¾	4.31	4.7	2.1	1.04	1.21	3.1	1.5	0.85	0.96	0.62
	⅝	3.67	4.1	1.8	1.06	1.17	2.8	1.3	0.87	0.92	0.62
3½ × 3	½	3.00	3.5	1.5	1.07	1.13	2.3	1.1	0.88	0.88	0.62
	⅜	2.30	2.7	1.1	1.09	1.08	1.8	0.85	0.90	0.83	0.62
	½	2.50	2.1	1.0	0.91	1.00	1.3	0.74	0.72	0.75	0.52
3 × 2½	⅜	1.92	1.7	0.81	0.93	0.96	1.0	0.58	0.74	0.71	0.52
	¼	1.31	1.2	0.56	0.95	0.91	0.74	0.40	0.75	0.66	0.53
	½	2.00	1.1	0.70	0.75	0.88	0.64	0.46	0.56	0.63	0.42
	⅜	1.55	0.91	0.55	0.77	0.83	0.51	0.36	0.58	0.58	0.42
2½ × 2	¼	1.06	0.65	0.38	0.78	0.79	0.37	0.25	0.59	0.54	0.42

TABLE XIV.—PROPERTIES OF NUMBERS

r	r^2	r^3	$\sqrt{r}$	$\sqrt[3]{r}$	$\frac{\pi r^2}{4}$	r	r^2	r^3	$\sqrt{r}$
1	1	1	1.0000	1.0000	0.7854	51	2601	132651	7.1414
2	4	8	1.4142	1.2599	3.1416	52	2704	140608	7.2111
3	9	27	1.7321	1.4422	7.0686	53	2809	148877	7.2801
4	16	64	2.0000	1.5874	12.5664	54	2916	157464	7.3485
5	25	125	2.2361	1.7100	19.6350	55	3025	166375	7.4162
6	36	216	2.4495	1.8171	28.2743	56	3136	175616	7.4833
7	49	343	2.6458	1.9129	38.4845	57	3249	185193	7.5498
8	64	512	2.8284	2.0000	50.2655	58	3364	195112	7.6158
9	81	729	3.0000	2.0801	63.6173	59	3481	205379	7.6811
10	100	1000	3.1623	2.1544	78.5398	60	3600	216000	7.7460
11	121	1331	3.3166	2.2240	95.0332	61	3721	226981	7.8102
12	144	1728	3.4641	2.2894	113.097	62	3844	238328	7.8740
13	169	2197	3.6056	2.3513	132.732	63	3969	250047	7.9373
14	196	2744	3.7417	2.4101	153.938	64	4096	262144	8.0000
15	225	3375	3.8730	2.4662	176.715	65	4225	274625	8.0623
16	256	4096	4.0000	2.5198	201.062	66	4356	287496	8.1240
17	289	4913	4.1231	2.5713	226.980	67	4489	300763	8.1854
18	324	5832	4.2426	2.6207	254.469	68	4624	314432	8.2462
19	361	6859	4.3589	2.6684	283.529	69	4761	328509	8.3066
20	400	8000	4.4721	2.7144	314.159	70	4900	343000	8.3666
21	441	9261	4.5826	2.7589	346.361	71	5041	357911	8.4261
22	484	10648	4.6904	2.8020	380.133	72	5184	373248	8.4853
23	529	12167	4.7958	2.8439	415.476	73	5329	389017	8.5440
24	576	13824	4.8990	2.8845	452.389	74	5476	405224	8.6023
25	625	15625	5.0000	2.9240	490.874	75	5625	421875	8.6603
26	676	17576	5.0990	2.9625	530.929	76	5776	438976	8.7178
27	729	19683	5.1962	3.0000	572.555	77	5929	456533	8.7750
28	784	21952	5.2915	3.0366	615.752	78	6084	474552	8.8318
29	841	24389	5.3852	3.0723	660.520	79	6241	493039	8.8882
30	900	27000	5.4772	3.1072	706.858	80	6400	512000	8.9443
31	961	29791	5.5678	3.1414	754.768	81	6561	531441	9.0000
32	1024	32768	5.6569	3.1748	804.248	82	6724	551368	9.0551
33	1089	35937	5.7446	3.2075	855.299	83	6889	571787	9.1104
34	1156	39304	5.8310	3.2396	907.920	84	7056	592704	9.1652
35	1225	42875	5.9161	3.2711	962.113	85	7225	614125	9.2195
36	1296	46656	6.0000	3.3019	1017.88	86	7396	636056	9.2736
37	1369	50653	6.0828	3.3322	1075.21	87	7569	658503	9.3274
38	1444	54872	6.1644	3.3620	1134.11	88	7744	681472	9.3808
39	1521	59319	6.2450	3.3912	1194.59	89	7921	704969	9.4340
40	1600	64000	6.3246	3.4200	1256.64	90	8100	729000	9.4868
41	1681	68921	6.4031	3.4482	1320.25	91	8281	753571	9.5394
42	1764	74088	6.4807	3.4760	1385.44	92	8464	778688	9.5917
43	1849	79507	6.5574	3.5034	1452.20	93	8649	804357	9.6437
44	1936	85184	6.6332	3.5303	1520.53	94	8836	830584	9.6954
45	2025	91125	6.7082	3.5569	1590.43	95	9025	857375	9.7468
46	2116	97336	6.7823	3.5830	1661.90	96	9216	884736	9.7980
47	2209	103823	6.8557	3.6088	1734.94	97	9409	912673	9.8489
48	2304	110592	6.9282	3.6342	1809.56	98	9604	941192	9.8995
49	2401	117649	7.0000	3.6593	1885.74	99	9801	970299	9.9499
50	2500	125000	7.0711	3.6840	1963.50	100	10000	1000000	10.0000

TABLE XIV—*continued*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
101	10201	1030301	10.0499	151	22801	3442951	12.2882
102	10404	1061208	10.0995	152	23104	3511808	12.3288
103	10609	1092727	10.1489	153	23409	3581577	12.3693
104	10816	1124864	10.1980	154	23716	3652264	12.4097
105	11025	1157625	10.2470	155	24025	3723875	12.4499
106	11236	1191016	10.2956	156	24336	3796416	12.4900
107	11449	1225043	10.3441	157	24649	3869893	12.5300
108	11664	1259712	10.3923	158	24964	3944312	12.5698
109	11881	1295029	10.4403	159	25281	4019679	12.6095
110	12100	1331000	10.4881	160	25600	4096000	12.6491
111	12321	1367631	10.5357	161	25921	4173281	12.6886
112	12544	1404928	10.5830	162	26244	4251528	12.7279
113	12769	1442897	10.6301	163	26569	4330747	12.7671
114	12996	1481544	10.6771	164	26896	4410944	12.8062
115	13225	1520875	10.7238	165	27225	4492125	12.8452
116	13456	1560896	10.7703	166	27556	4574296	12.8841
117	13689	1601613	10.8167	167	27889	4657463	12.9228
118	13924	1643032	10.8628	168	28224	4741632	12.9615
119	14161	1685159	10.9087	169	28561	4826809	13.0000
120	14400	1728000	10.9545	170	28900	4913000	13.0384
121	14641	1771561	11.0000	171	29241	5000211	13.0767
122	14884	1815848	11.0454	172	29584	5088448	13.1149
123	15129	1860867	11.0905	173	29929	5177717	13.1529
124	15376	1906624	11.1355	174	30276	5268024	13.1909
125	15625	1953125	11.1803	175	30625	5359375	13.2288
126	15876	2000376	11.2250	176	30976	5451776	13.2665
127	16129	2048383	11.2694	177	31329	5545233	13.3041
128	16384	2097152	11.3137	178	31684	5639752	13.3417
129	16641	2146689	11.3578	179	32041	5735339	13.3791
130	16900	2197000	11.4018	180	32400	5832000	13.4164
131	17161	2248091	11.4455	181	32761	5929741	13.4536
132	17424	2299968	11.4891	182	33124	6028568	12.4907
133	17689	2352637	11.5326	183	33489	6128487	13.5277
134	17956	2406104	11.5758	184	33856	6229504	13.5647
135	18225	2460375	11.6190	185	34225	6331625	12.6015
136	18496	2515456	11.6619	186	34596	6434856	13.6382
137	18769	2571353	11.7047	187	34969	6539203	13.6748
138	19044	2628072	11.7473	188	35344	6644672	13.7113
139	19321	2685619	11.7898	189	35721	6751269	13.7477
140	19600	2744000	11.8322	190	36100	6859000	13.7840
141	19881	2803221	11.8743	191	36481	6967871	13.8203
142	20164	2863288	11.9164	192	36864	7077888	13.8564
143	20449	2924207	11.9583	193	37249	7189057	13.8924
144	20736	2985984	12.0000	194	37636	7301384	13.9284
145	21025	3048625	12.0416	195	38025	7414875	13.9642
146	21316	3112136	12.0830	196	38416	7529536	14.0000
147	21609	3176523	12.1244	197	38809	7645373	14.0357
148	21904	3241792	12.1655	198	39204	7762392	14.0712
149	22201	3307949	12.2066	199	39601	7880599	14.1067
150	22500	3375000	12.2474	200	40000	8000000	14.1421

TABLE XIV—*continued*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
201	40401	8120601	14.1774	251	63001	15813251	15.8430
202	40804	8242408	14.2127	252	63504	16003008	15.8745
203	41209	8365427	14.2478	253	64009	16194277	15.9060
204	41616	8489664	14.2829	254	64516	16387064	15.9374
205	42025	8615125	14.3178	255	65025	16581375	15.9687
206	42436	8741816	14.3527	256	65536	16777216	16.0000
207	42849	8869743	14.3875	257	66049	16974593	16.0312
208	43264	8998912	14.4222	258	66564	17173512	16.0624
209	43681	9129329	14.4568	259	67081	17373979	16.0935
210	44100	9261000	14.9414	260	67600	17576000	16.1245
211	44521	9393931	14.5258	261	68121	17779581	16.1555
212	44944	9528128	14.5602	262	68644	17984728	16.1864
213	45369	9663597	14.5945	263	69169	18191447	16.2173
214	45796	9800344	14.6287	264	69696	18399744	16.2481
215	46225	9938375	14.6629	265	70225	18609625	16.2788
216	46656	10077696	14.6969	266	70756	18821096	16.3095
217	47089	10218313	14.7309	267	71289	19034163	16.3401
218	47524	10360232	14.7648	268	71824	19248832	16.3707
219	47961	10503459	14.7986	269	72361	19465109	16.4012
220	48400	10648000	14.8324	270	72900	19683000	16.4317
221	48841	10793861	14.8661	271	73441	19902511	16.4621
222	49284	10941048	14.8997	272	73984	20123648	16.4924
223	49729	11089567	14.9332	273	74529	20346417	16.5227
224	50176	11239424	14.9666	274	75076	20570824	16.5529
225	50625	11390625	15.0000	275	75625	20796875	16.5831
226	51076	11543176	15.0333	276	76176	21024576	16.6132
227	51529	11697083	15.0665	277	76729	21253933	16.6433
228	51984	11852352	15.0997	278	77284	21484952	16.6733
229	52441	12008989	15.1327	279	77841	21717639	16.7033
230	52900	12167000	15.1658	280	78400	21952000	16.7332
231	53361	12326391	15.1987	281	78961	22188041	16.7631
232	53824	12487168	15.2315	282	79524	22425768	16.7929
233	54289	12649337	15.2643	283	80089	22665187	16.8226
234	54756	12812904	15.2971	284	80656	22906304	16.8523
235	55225	12977875	15.3297	285	81225	23149125	16.8819
236	55696	13144256	15.3623	286	81796	23393656	16.9115
237	56169	13312053	15.3948	287	82369	23639903	16.9411
238	56644	13481272	15.4272	288	82944	23887872	16.9706
239	57121	13651919	15.4596	289	83521	24137569	17.0000
240	57600	13824000	15.4919	290	84100	24389000	17.0294
241	58081	13997521	15.5242	291	84681	24642171	17.0587
242	58564	14172488	15.5563	292	85264	24897088	17.0880
243	59049	14348907	15.5885	293	85849	25153757	17.1172
244	59536	14526784	15.6205	294	86436	25412184	17.1464
245	60025	14706125	15.6525	295	87025	25672375	17.1756
246	60516	14886936	15.6844	296	87616	25934336	17.2047
247	61009	15069223	15.7162	297	88209	26198073	17.2337
248	61504	15252992	15.7480	298	88804	26463592	17.2627
249	62001	15438249	15.7797	299	89401	26730899	17.2916
250	62500	15625000	15.8114	300	90000	27000000	17.3205

TABLE XIV—*continued*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
301	90601	27270901	17.3494	351	123201	43243551	18.7350
302	91204	27543608	17.3781	352	123904	43614208	18.7617
303	91809	27818127	17.4069	353	124609	43986977	18.7883
304	92416	28094464	17.4356	354	125316	44361864	18.8149
305	93025	28372625	17.4642	355	126025	44738875	18.8414
306	93636	28652616	17.4929	356	126736	45118016	18.8680
307	94249	28934443	17.5214	357	127449	45499293	18.8944
308	94864	29218112	17.5499	358	128164	45882712	18.9209
309	95481	29503629	17.5784	359	128881	46268279	18.9473
310	96100	29791000	17.6068	360	129600	46656000	18.9737
311	96721	30080231	17.6352	361	130321	47045881	19.0000
312	97344	30371328	17.6635	362	131044	47437928	19.0263
313	97969	30664297	17.6918	363	131769	47832147	19.0526
314	98596	30959144	17.7200	364	132496	48228544	19.0788
315	99225	31255875	17.7482	365	133225	48627125	19.1050
316	99856	31554496	17.7764	366	133956	49027896	19.1311
317	100489	31855013	17.8045	367	134689	49430863	19.1572
318	101124	32157432	17.8326	368	135424	49836032	19.1833
319	101761	32461759	17.8606	369	166161	50243409	19.2094
320	102400	32768000	17.8885	370	136900	50653000	19.2354
321	103041	33076161	17.9165	371	137641	51064811	19.2614
322	103684	33386248	17.9444	372	138384	51478848	19.2873
323	104329	33698267	17.9722	373	139129	51895117	19.3132
324	104976	34012224	18.0000	374	139876	52313624	19.3391
325	105625	34328125	18.0278	375	140625	52734375	19.3649
326	106276	34645976	18.0555	376	141376	53157376	19.3907
327	106929	34965783	18.0831	377	142129	53582633	19.4165
328	107584	35287552	18.1108	378	142884	54010152	19.4422
329	108241	35611289	18.1384	379	143641	54439939	19.4679
330	108900	35937000	18.1659	380	144400	54872000	19.4936
331	109561	36264691	18.1934	381	145161	55306341	19.5192
332	110224	36594368	18.2209	382	145024	55742968	19.5448
333	110889	36926037	18.2483	383	146689	56181887	19.5704
334	111556	37259704	18.2757	384	147456	56623104	19.5959
335	112225	37595375	18.3030	385	148225	57066625	19.6214
336	112896	37933056	18.3303	386	148996	57512456	19.6469
337	113569	38272753	18.3576	387	149769	57960603	19.6723
338	114244	38614472	18.3848	388	150544	58411072	19.6977
339	114921	38958219	18.4120	389	151321	58863869	19.7231
340	115600	39304000	18.4391	390	152100	59319000	19.7484
341	116281	39651821	18.4662	391	152881	59776471	19.7737
342	116964	40001688	18.4932	392	153664	60236288	19.7990
343	117649	40353607	18.5203	393	154449	60698457	19.8242
344	118336	40707584	18.5472	394	155236	61162984	19.8494
345	119025	41063625	18.5742	395	156025	61629875	19.8746
346	119716	41421736	18.6011	396	156816	62099136	19.8997
347	120409	41781923	18.6279	397	157609	62570773	19.9249
348	121104	42144192	18.6548	398	158404	63044792	19.9499
349	121801	42508549	18.6815	399	159201	63521199	19.9750
350	122500	42875000	18.7083	400	160000	64000000	20.0000

TABLE XIV—*continued*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
401	160801	64481201	20.0250	451	203401	91733851	21.2368
402	161604	64964808	20.0499	452	204304	92345408	21.2603
403	162409	65450827	20.0749	453	205209	92959677	21.2838
404	163216	65939264	20.0998	454	206116	93576664	21.3073
405	164025	66430125	20.1246	455	207025	94196375	21.3307
406	164836	66923416	20.1494	456	207936	94818816	21.3542
407	165649	67419143	20.1742	457	208849	95443993	21.3776
408	166464	67917312	20.1990	458	209764	96071912	21.4009
409	167281	68417929	20.2237	459	210681	96702579	21.4243
410	168100	68921000	20.2485	460	211600	97336000	21.4476
411	168921	69426531	20.2731	461	212521	97972181	21.4709
412	169744	69934528	20.2978	462	213444	98611128	21.4942
413	170569	70444997	20.3224	463	214369	99252847	21.5174
414	171396	70957944	20.3470	464	215296	99897344	21.5407
415	172225	71473375	20.3715	465	216225	100544625	21.5639
416	173056	71991296	20.3961	466	217156	101194696	21.5870
417	173889	72511713	20.4206	467	218089	101847563	21.6102
418	174724	73034632	20.4450	468	219024	102503232	21.6333
419	175561	73560059	20.4695	469	219961	103161709	21.6564
420	176400	74088000	20.4939	470	220900	103823000	21.6795
421	177241	74618461	20.5183	471	221841	104487111	21.7025
422	178084	75151448	20.5426	472	222784	105154048	21.7256
423	178929	75686967	20.5670	473	223729	105823817	21.7486
424	179776	76225024	20.5913	474	224676	106496424	21.7715
425	180625	76765625	20.6155	475	225625	107171875	21.7945
426	181476	77308776	20.6398	476	225676	107850176	21.8174
427	182329	77854483	20.6640	477	227529	108531333	21.8403
428	183184	78402752	20.6882	478	228484	109215352	21.8632
429	184041	78953589	20.7123	479	229441	109902239	21.8861
430	184900	79507000	20.7364	480	230400	110592000	21.9089
431	185761	80062991	20.7605	481	231361	111284641	21.9317
432	186624	80621568	20.7846	482	232324	111980168	21.9545
433	187489	81182737	20.8087	483	233289	112678587	21.9773
434	188356	81746504	20.8327	484	234256	113379904	22.0000
435	189225	82312875	20.8567	485	235225	114084125	22.0227
436	190096	82881856	20.8806	486	236196	114791256	22.0454
437	190969	83453453	20.9045	487	237169	115501303	22.0681
438	191844	84027672	20.9284	488	238144	116214272	22.0907
439	192721	84604519	20.9523	489	239121	116930169	22.1133
440	193600	85184000	20.9762	490	240100	117649000	22.1359
441	194481	85766121	21.0000	491	241081	118370771	22.1585
442	195364	86350888	21.0238	492	242064	119095488	22.1811
443	196249	86938307	21.0476	493	243049	119823157	22.2036
444	197136	87528384	21.0713	494	244036	120553784	22.2261
445	198025	88121125	21.0950	495	235025	121287375	22.2486
446	198916	88716536	21.1187	496	246016	122023936	22.2711
447	199809	89314623	21.1424	497	247009	122763473	22.2935
448	200704	89915392	21.1660	498	248004	123505992	22.3159
449	201601	90518849	21.1896	499	249001	124251499	22.3383
450	202500	91125000	21.2132	500	250000	125000000	22.3607

TABLE XIV—*continued*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
501	251001	125751501	22.3830	551	303601	167284151	23.4734
502	252004	126506008	22.4054	552	304704	168196608	23.4947
503	253009	127263527	22.4277	553	305809	169112377	23.5160
504	254016	128024064	22.4499	554	306916	170031464	23.5372
505	255025	128787625	22.4722	555	308025	170953875	23.5584
506	256036	129554216	22.4944	556	309136	171879616	23.5797
507	257049	130323843	22.5167	557	310249	172808693	23.6008
508	258064	131096512	22.5389	558	311364	173741112	23.6220
509	259081	131872229	22.5610	559	312481	174676879	23.6432
510	260100	132651000	22.5832	560	313600	175616000	23.6643
511	261121	133432831	22.6053	561	314721	176558481	23.6854
512	262144	134217728	22.6274	562	315844	177504328	23.7065
513	263169	135005697	22.6495	563	316969	178453547	23.7276
514	264196	135796744	22.6716	564	318096	179406144	23.7487
515	265225	136590875	22.6936	565	319225	180362125	23.7697
516	266256	137388096	22.7156	566	320356	181321496	23.7908
517	267289	138188413	22.7376	567	321489	182284263	23.8118
518	268324	138991832	22.7596	568	322624	183250432	23.8328
519	269361	139798359	22.7816	569	323761	184220009	23.8537
520	270400	140608000	22.8035	570	324900	185193000	23.8747
521	271441	141420761	22.8254	571	326041	186169411	23.8956
522	272484	142236648	22.8473	572	327184	187149248	23.9165
523	273529	143055667	22.8692	573	328329	188132517	23.9374
524	274576	143877824	22.8910	574	329476	189119224	23.9583
525	275625	144703125	22.9129	575	330625	190109375	23.9792
526	276676	145531576	22.9347	576	331776	191102976	24.0000
527	277729	146363183	22.9565	577	332929	192100033	24.0208
528	278784	147197952	22.9783	578	334084	193100552	24.0416
529	279841	148035889	23.0000	579	335241	194104539	24.0624
530	280900	148877000	23.0217	580	336400	195112000	24.0832
531	281961	149721291	23.0434	581	337561	196122941	24.1039
532	283024	150568768	23.0651	582	338724	197137368	24.1247
533	284089	151419437	23.0868	583	339889	198155287	24.1454
534	285156	152273304	23.1084	584	341056	199176704	24.1661
535	286225	153130375	23.1301	585	342225	200201625	24.1868
536	287296	153990656	23.1517	586	343396	201230056	24.2074
537	288369	154854153	23.1733	587	344569	202262003	24.2281
538	289444	155720872	23.1948	588	345744	203297472	24.2487
539	290521	156590819	23.2164	589	346921	204336469	24.2693
540	291600	157464000	23.2379	590	348100	205379000	24.2899
541	292681	158340421	23.2594	591	349281	206425071	24.3105
542	293764	159220088	23.2809	592	350464	207474688	24.3311
543	294849	160103007	23.3034	593	351649	208527857	24.3516
544	295936	160989184	23.3238	594	352836	209584584	24.3721
545	297025	161878625	23.3452	595	354925	210644875	24.3926
546	298116	162771336	23.3666	596	355216	211708736	24.4131
547	299209	163667323	23.3880	597	356409	212776173	24.4336
548	300304	164566592	23.4094	598	357604	213847192	24.4540
549	301401	165469149	23.4307	599	358801	214921799	24.4745
550	302500	166375000	23.4521	600	360000	216000000	24.4949

TABLE XIV—*continued*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
601	361201	217081801	24.5153	651	423801	275894451	25.5147
602	362404	218167208	24.5357	652	425104	277167808	25.5343
603	363609	219256227	24.5561	653	426409	278445077	25.5539
604	364816	220348864	24.5764	654	427716	279726264	25.5734
605	366025	221445125	24.5967	655	429025	281011375	25.5930
606	367236	222545016	24.6171	656	430336	282300416	25.6125
607	368449	223648543	24.6374	657	431649	283593393	25.6320
608	369664	224755712	24.6577	658	432964	284890312	25.6515
609	370881	225866529	24.6779	659	434281	286191179	25.6710
610	372100	226981000	24.6982	660	435600	287496000	25.6905
611	373321	228099131	24.7184	661	436921	288804781	25.7099
612	374544	229220928	24.7386	662	438244	290117528	25.7294
613	375769	230346397	24.7588	663	439569	291434247	25.7488
614	376996	231475544	24.7790	664	440896	292754944	25.7682
615	378225	232608375	24.7992	665	442225	294079625	25.7876
616	379456	233744896	24.8193	666	443556	295408296	25.8070
617	380689	234885113	24.8395	667	444889	296740963	25.8263
618	381924	236029032	24.8596	668	446224	298077632	25.8457
619	383161	237176659	24.8797	669	447561	299418309	25.8650
620	384400	238328000	24.8998	670	448900	300763000	25.8844
621	385641	239483061	24.9199	671	450241	302111711	25.9037
622	386884	240641848	24.9399	672	451584	303464448	25.9230
623	388129	241804367	24.9600	673	452929	304821217	25.9422
624	389376	242970624	24.9800	674	454276	306182024	25.9615
625	390625	244140625	25.0000	675	455625	307546875	25.9808
626	391876	245314376	25.0200	676	456976	308915776	26.0000
627	393129	246491883	25.0400	677	458329	310288733	26.0192
628	394384	247673152	25.0599	678	459684	311665752	26.0384
629	395641	248858189	25.0799	679	461041	313046839	26.0576
630	396900	250047000	25.0998	680	462400	314432000	26.0768
631	398161	251239591	25.1197	681	463761	315821241	26.0960
632	399424	252435968	25.1396	682	465124	317214568	26.1151
633	400689	253636137	25.1595	683	466489	318611987	26.1343
634	401956	254840104	25.1794	684	467856	320013504	26.1534
635	403225	256047875	25.1992	685	469225	321419125	26.1725
636	404496	257259456	25.2190	686	470596	322828856	26.1916
637	405769	258474853	25.2389	687	471969	324242703	26.2107
638	407044	259694072	25.2587	688	473344	325660672	26.2298
639	408321	260917119	25.2784	689	474721	327082769	26.2488
640	409600	262144000	25.2982	690	476100	328509000	26.2679
641	410881	263374721	25.3180	691	477481	329939371	26.2869
642	412164	264609288	25.3377	692	478864	331373888	26.3059
643	413449	265847707	25.3574	693	480249	332812557	26.3249
644	414736	267089984	25.3772	694	481636	334255384	26.3439
645	416025	268336125	25.3969	695	483025	335702375	26.3629
646	417316	269586136	25.4165	696	484416	337153536	26.3818
647	418609	270840023	25.4362	697	485809	338608873	26.4008
648	419904	272097792	25.4558	698	487204	340068392	26.4197
649	421201	273359449	25.4755	699	488601	341532099	26.4386
650	422500	274625000	25.4951	700	490000	343000000	26.4575

TABLE XIV—*continued*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
701	491401	244472101	26.4764	751	564001	423564751	27.4044
702	492804	345948408	26.4953	752	565504	425259008	27.4226
703	494209	347428927	26.5141	753	567009	426957777	27.4408
704	495616	348913664	26.5330	754	568516	428661064	27.4591
705	497025	350402625	26.5518	755	570025	430368875	27.4773
706	498436	351895816	26.5707	756	571536	432081216	27.4955
707	499849	353393243	26.5895	757	573049	433798093	27.5136
708	501264	354894912	26.6083	758	574564	435519512	25.5318
709	502681	356400829	26.6271	759	576081	437245479	27.5500
710	504100	357911000	26.6458	760	577600	438976000	27.5681
711	505521	359425431	26.6646	761	579121	440711081	27.5862
712	506044	360944128	26.6833	762	580644	442450728	27.6043
713	508369	362467097	26.7021	763	582169	444194947	27.6225
714	509796	363994344	26.7208	764	583696	445943744	27.6405
715	511225	365525875	26.7395	765	585225	447697125	27.6586
716	512656	367061696	26.7582	766	586756	449455096	27.6767
717	514089	368601813	26.7769	767	588289	451217663	27.6948
718	515524	370146232	26.7955	768	589824	452984832	27.7128
719	516961	371694959	26.8142	769	591361	454756609	27.7308
720	518400	373248000	26.8328	770	592900	456533000	27.7489
721	519841	374805361	26.8514	771	594441	458314011	27.7669
722	521284	376367048	26.8701	772	595984	460099648	27.7849
723	522729	377933067	26.8887	773	597529	461889917	27.8029
724	524176	379503424	26.9072	774	599076	463684824	27.8209
725	525625	381078125	26.9258	775	600625	465484375	27.8388
726	527076	382657176	26.9444	776	602176	467288576	27.8568
727	528529	384240583	26.9629	777	603729	469097433	27.8747
728	529984	385828352	26.9815	778	605284	470910952	27.8927
729	531441	387420489	27.0000	779	606841	472729139	27.9106
730	532900	389017000	27.0185	780	608400	474552000	27.9285
731	534361	390717891	27.0370	781	609961	476379541	27.9464
732	535824	392223168	27.0555	782	611524	478211768	27.9643
733	537289	393832837	27.0740	783	613089	480048687	27.9821
734	538756	395446904	27.0924	784	614656	481890304	28.0000
735	540225	397065375	27.1109	785	616225	483736625	28.0179
736	541696	398688256	27.1293	786	617796	485587756	28.0357
737	543169	400315553	27.1477	787	619369	487443403	28.0535
738	544644	401947272	27.1662	788	620944	489303872	28.0713
730	546121	403583419	27.1846	789	622521	491169069	28.0891
740	547600	405224000	27.2029	790	624100	493039000	28.1069
741	549081	406869021	27.2213	791	625681	494913671	28.1247
742	550564	408518488	27.2397	792	627264	496793088	28.1425
743	552049	410172407	27.2580	793	628849	498677257	28.1603
744	553536	411830784	27.2764	794	630436	500566184	28.1780
745	555025	413403625	27.2947	795	632025	502459875	28.1957
746	556516	415160936	27.3130	796	633616	504358336	28.2135
747	558009	416832723	27.3313	797	635209	506261573	28.2312
748	559504	418508992	27.3496	798	636804	508169592	28.2489
749	561001	420189749	27.3679	799	638401	510082399	28.2666
750	562500	421875000	27.3861	800	640000	512000000	28.2843

TABLE XIV—*continued*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
801	641601	513922401	28.3019	851	724201	616295051	29.1719
802	643204	515849608	28.3196	852	725904	617470208	29.1890
803	644809	517781627	28.3373	853	727609	620650477	29.2063
804	646416	519718464	28.3549	854	729316	622835864	29.2233
805	648025	521660125	28.3725	855	731025	625026375	29.2404
806	649636	523606616	28.3901	856	732736	627222016	29.2575
807	651249	525557943	28.4077	857	734449	629422793	29.2746
808	652864	527514112	28.4253	858	736164	631628712	29.2916
809	654481	529475129	28.4429	859	737881	633839779	29.3087
810	656100	531441000	28.4605	860	739600	636056000	29.3258
811	657721	533411731	28.4781	861	741321	638277381	29.3428
812	659344	535387328	28.4956	862	743044	640503928	29.3598
813	660969	537367797	28.5132	863	744769	642735647	29.3769
814	662596	539353144	28.5307	864	746796	644972544	29.3939
815	664225	541343375	28.5482	865	748225	647214625	29.4109
816	665856	543338496	28.5657	866	749956	649461896	29.4279
817	667489	545338513	28.5832	867	751689	651714363	29.4449
818	669124	547343432	28.6007	868	753424	653972032	29.4618
819	670761	549353259	28.6182	869	755161	656234909	29.4788
820	672400	551368000	28.6356	870	756900	658503000	29.4958
821	674041	553387661	28.6531	871	758641	660776311	29.5127
822	678684	555412248	28.6705	872	760384	663054848	29.5296
823	677329	557441767	28.6880	873	762129	665338617	29.5466
824	678976	559476224	28.7054	874	763876	667627624	29.5635
825	680625	561515625	28.7228	875	765625	669921875	20.5804
826	682276	563559976	28.7402	876	767376	672221376	29.5973
827	683929	565609283	28.7576	877	769129	674526133	29.6142
828	685584	567663552	28.7750	878	770884	676836152	29.6311
829	687241	569722789	28.7924	879	772641	679151439	29.6479
830	688900	571787000	28.8097	880	774400	681472000	29.6648
831	690561	573856191	28.8271	881	776161	683797841	29.6816
832	692224	575930368	28.8444	882	777924	686128968	29.6985
833	693889	578009537	28.8617	883	779689	688465387	29.7153
834	695556	580093704	28.8791	884	781456	690807104	29.7321
835	697225	582182875	28.8964	885	783225	693154125	29.7489
836	698896	584277056	28.9137	886	784996	695506456	29.7658
837	700569	586376253	28.9310	887	786769	697864130	29.7825
838	702244	588480472	28.9482	888	788544	700227072	29.7993
839	703921	590589719	28.9655	889	790321	702595369	29.8161
840	705600	592704000	28.9828	890	792100	704969000	29.8329
841	707281	594823321	29.0000	891	793881	707347971	29.8496
842	708964	596947688	29.0172	892	795664	709732288	29.8664
843	710649	599077107	29.0345	893	797449	712121957	29.8831
844	712336	601211584	29.0517	894	799236	714516984	29.8998
845	714025	603351125	29.0689	895	801025	716917375	29.9166
846	715716	605495736	29.0861	896	802816	719323136	29.9333
847	717409	607645423	29.1033	897	804609	721734273	29.9500
848	719104	609800192	29.1204	898	806404	724150792	29.9666
849	720801	611960049	29.1376	899	808201	726572699	29.9833
850	722500	614125000	29.1548	900	810000	729000000	30.0000

Table XIV—*concluded*

r	r^2	r^3	$\sqrt{r}$	r	r^2	r^3	$\sqrt{r}$
901	811801	731432701	30.0167	951	904401	860085351	30.8383
902	813604	733870808	30.0333	952	906304	862801408	30.8545
903	815409	736314327	30.0500	953	908209	865523177	30.8707
904	817216	738763264	30.0666	954	910116	868250664	30.8869
905	819025	741217625	30.0832	955	912025	870983875	30.9031
906	820836	743677416	30.0998	956	913936	873722816	30.9192
907	822649	746142643	30.1164	957	915849	876467493	30.9354
908	824464	748613312	30.1330	958	917764	879217912	30.9516
909	826281	751089429	30.1496	959	919681	881974079	30.9677
910	828100	753571000	30.1662	960	921600	884736000	30.9839
911	829921	756058031	30.1828	961	923521	887503681	31.0000
912	831744	758550528	30.1993	962	925444	890277128	31.0161
913	833569	761048497	30.2159	963	927369	893056347	31.0322
914	835396	763551944	30.2324	964	929296	895841344	31.0483
915	837225	766060875	30.2490	965	931225	898632125	31.0644
916	839056	768575296	30.2655	966	933156	901428696	31.0805
917	840889	771095213	30.2820	967	935089	904231063	31.0966
918	842724	773620632	30.2985	968	937024	907039232	31.1127
919	844561	776151559	30.3150	969	938961	909853209	31.1288
920	846400	778688000	30.3315	970	940900	912673000	31.1448
921	848241	781229961	30.3480	971	942841	915498611	31.1609
922	850084	783777448	30.3645	972	944784	918330048	31.1769
923	851929	786330467	30.3809	973	946729	921167317	31.1929
924	853776	788889024	30.3974	974	948676	924010424	31.2090
925	855625	791453125	30.4138	975	950625	926859375	31.2250
926	857476	794022776	30.4302	976	952576	929714176	31.2410
927	859329	796597983	30.4467	977	954529	932574833	31.2570
928	861184	799178752	30.4631	978	956484	935441352	31.2730
929	863041	801765089	30.4795	979	958441	938313739	31.2890
930	864900	804357000	30.4959	980	960400	941192000	31.3050
931	866761	806954491	30.5123	981	962361	944076141	31.3209
932	868624	809557568	30.5287	982	964324	946966168	31.3369
933	870489	812166237	30.5450	983	966289	949862087	31.3528
934	872356	814780504	30.5614	984	968256	952763904	31.3688
935	874225	817400375	30.5778	985	970225	955671625	31.3847
936	876096	820025856	30.5941	986	972196	958585256	31.4006
937	877969	822656953	30.6105	987	974169	961504803	31.4166
938	879844	825293672	30.6268	988	976144	964430272	31.4325
939	881721	827936019	30.6431	989	978121	967361669	31.4484
940	883600	830584000	30.6594	990	980100	970299000	31.4643
941	885481	833237621	30.6757	991	982081	973242271	31.4802
942	887364	835896888	30.6920	992	984064	976191488	31.4960
943	889249	838561807	30.7083	993	986049	979146657	31.5119
944	891136	841232384	30.7246	994	988036	982107784	31.5278
945	893025	843908625	30.7409	995	990025	985074875	31.5436
946	894916	846590536	30.7571	996	992016	988047936	31.5595
947	896809	849278123	30.7734	997	994009	991026973	31.5753
948	898704	851971392	30.7896	998	996004	994011992	31.5911
949	900601	854670349	30.8058	999	998001	997002999	31.6070
950	902500	857375000	30.8221	1000	1000000	1000000000	31.6228

TABLE XV.—COMMON LOGARITHMS

N.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N.	0	1	2	3	4	5	6	7	8	9

TABLE XV—*concluded*

N.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N.	0	1	2	3	4	5	6	7	8	9

TABLE XVI.—TRIGONOMETRIC FUNCTIONS

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
0° 00′	.0000	∞	1 0000	0.0000	.0000	∞	∞	∞	90° 00′
10	.0029	7.4637	1.0000	0000	.0029	7.4637	343.77	2.5363	50
20	.0058	7642	1.0000	0000	.0058	7648	171.89	2352	40
30	.0087	9408	1.0000	0000	.0087	9409	114.59	0591	30
40	.0116	8.0658	.9999	0000	.0116	8.0658	85.940	1.9342	20
50	.0145	1627	.9999	0000	.0145	1627	68.750	8373	10
1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.290	1.7581	89° 00′
10	.0204	3088	.9998	9999	.0204	3089	49.104	6911	50
20	.0233	3668	.9997	9999	.0233	3669	42.964	6331	40
30	.0262	4179	.9997	9999	.0262	4181	38.188	5819	30
40	.0291	4637	.9996	9998	.0291	4638	34.368	5362	20
50	.0320	5050	.9995	9998	.0320	5053	31.242	4947	10
2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.636	1.4569	88° 00′
10	.0378	5776	.9993	9997	.0378	5779	26.432	4221	50
20	.0407	6097	.9992	9996	.0407	6101	24.542	3899	40
30	.0436	6397	.9990	9996	.0437	6401	22.904	3599	30
40	.0465	6677	.9989	9995	.0466	6682	21.470	3318	20
50	.0494	6940	.9988	9995	.0495	6945	20.206	3055	10
3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.081	1.2806	87° 00′
10	.0552	7423	.9985	9993	.0553	7429	18.075	2571	50
20	.0581	7645	.9983	9993	.0582	7652	17.169	2348	40
30	.0610	7857	.9981	9992	.0612	7865	16.350	2135	30
40	.0640	8059	.9980	9991	.0641	8067	15.605	1933	20
50	.0669	8251	.9978	9990	.0670	8261	14.924	1739	10
4° 00′	.0698	8.8436	.9976	9.9989	.0699	8.8446	14.301	1.1554	86° 00′
10	.0727	8613	.9974	9989	.0729	8624	13.727	1376	50
20	.0756	8783	.9971	9988	.0758	8795	13.197	1205	40
30	.0785	8946	.9969	9987	.0787	8960	12.706	1040	30
40	.0814	9104	.9967	9986	.0816	9118	12.251	0882	20
50	.0843	9256	.9964	9985	.0846	9272	11.826	0728	10
5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.430	1.0580	85° 00′
10	.0901	9545	.9959	9982	.0904	9563	11.059	0437	50
20	.0929	9682	.9957	9981	.0934	9701	10.712	0299	40
30	.0958	9816	.9954	9980	.0963	9836	10.385	0164	30
40	.0987	9945	.9951	9979	.0992	9966	10.078	0034	20
50	.1016	9.0070	.9948	9977	.1022	9.0093	9.7882	0.9907	10
6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.5144	0.9784	84° 00′
10	.1074	0311	.9942	9975	.1080	0336	9.2553	9664	50
20	.1103	0426	.9939	9973	.1110	0453	9.0098	9547	40
30	.1132	0539	.9936	9972	.1139	0567	8.7769	9433	30
40	.1161	0648	.9932	9971	.1169	0678	8.5555	9322	20
50	.1190	0755	.9929	9969	.1198	0786	8.3450	9214	10
7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.1443	0.9109	83° 00′
10	.1248	0961	.9922	9966	.1257	0995	7.9530	9005	50
20	.1276	1060	.9918	9964	.1287	1096	7.7704	8904	40
30	.1305	1157	.9914	9963	.1317	1194	7.5958	8806	30
40	.1334	1252	.9911	9961	.1346	1291	7.4287	8709	20
50	.1363	1345	.9907	9959	.1376	1385	7.2687	8615	10
8° 00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.1154	0.8522	82° 00′
10	.1421	1525	.9899	9956	.1435	1569	6.9682	8431	50
20	.1449	1612	.9894	9954	.1465	1658	6.8269	8342	40
30	.1478	1697	.9890	9952	.1495	1745	6.6912	8255	30
40	.1507	1781	.9886	9950	.1524	1831	6.5606	8169	20
50	.1536	1863	.9881	9948	.1554	1915	6.4348	8085	10
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE XVI—*continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
10	.1593	2022	.9872	9944	.1614	2078	6.1970	7922	50
20	.1622	2100	.9868	9942	.1644	2158	6.0844	7842	40
30	.1650	2176	.9863	9940	.1673	2236	5.9758	7764	30
40	.1679	2251	.9858	9938	.1703	2313	5.8708	7687	20
50	.1708	2324	.9853	9936	.1733	2389	5.7694	7611	10
10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.6713	0.7537	80° 00′
10	.1765	2468	.9843	9931	.1793	2536	5.5764	7464	50
20	.1794	2538	.9838	9929	.1823	2609	5.4845	7391	40
30	.1822	2606	.9833	9927	.1853	2680	5.3955	7320	30
40	.1851	2674	.9827	9924	.1883	2750	5.3093	7250	20
50	.1880	2740	.9822	9922	.1914	2819	5.2257	7181	10
11° 00′	.1908	9.2806	.9316	9.9919	.1944	9.2887	5.1446	0.7113	79° 00′
10	.1937	2870	.9811	9917	.1974	2953	5.0658	7047	50
20	.1965	2934	.9805	9914	.2004	3020	4.9894	6980	40
30	.1994	2997	.9799	9912	.2035	3085	4.9152	6915	30
40	.2022	3058	.9793	9909	.2065	3149	4.8430	6851	20
50	.2051	3119	.9787	9907	.2095	3212	4.7729	6788	10
12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.7046	0.6725	78° 00′
10	.2108	3238	.9775	9901	.2156	3336	4.6382	6664	50
20	.2136	3296	.9769	9899	.2186	3397	4.5736	6603	40
30	.2164	3353	.9763	9896	.2217	3458	4.5107	6542	30
40	.2193	3410	.9757	9893	.2247	3517	4.4494	6483	20
50	.2221	3466	.9750	9890	.2278	3576	4.3897	6424	10
13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.3315	0.6366	77° 00′
10	.2278	3575	.9737	9884	.2339	3691	4.2747	6309	50
20	.2306	3629	.9730	9881	.2370	3748	4.2193	6252	40
30	.2334	3682	.9724	9878	.2401	3804	4.1653	6196	30
40	.2363	3734	.9717	9875	.2432	3859	4.1126	6141	20
50	.2391	3786	.9710	9872	.2462	3914	4.0611	6086	10
14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.0108	0.6032	76° 00′
10	.2447	3887	.9696	9866	.2524	4021	3.9617	5979	50
20	.2476	3937	.9689	9863	.2555	4074	3.9136	5926	40
30	.2504	3986	.9681	9859	.2586	4127	3.8667	5873	30
40	.2532	4035	.9674	9856	.2617	4178	3.8208	5822	20
50	.2560	4083	.9667	9853	.2648	4230	3.7760	5770	10
15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.7321	0.5719	75° 00′
10	.2616	4177	.9652	9846	.2711	4331	3.6891	5669	50
20	.2644	4223	.9644	9843	.2742	4381	3.6470	5619	40
30	.2672	4269	.9636	9839	.2773	4430	3.6059	5570	30
40	.2700	4314	.9628	9836	.2805	4479	3.5656	5521	20
50	.2728	4359	.9621	9832	.2836	4527	3.5261	5478	10
16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.4874	0.5425	74° 00′
10	.2784	4447	.9605	9825	.2899	4622	3.4495	5378	50
20	.2812	4491	.9596	9821	.2931	4669	3.4124	5331	40
30	.2840	4533	.9588	9817	.2962	4716	3.3759	5284	30
40	.2868	4576	.9580	9814	.2994	4762	3.3402	5238	20
50	.2896	4618	.9572	9810	.3026	4808	3.3052	5192	10
17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.2709	0.5147	73° 00′
10	.2952	4700	.9555	9802	.3089	4898	3.2371	5102	50
20	.2979	4741	.9546	9798	.3121	4943	3.2041	5057	40
30	.3007	4781	.9537	9794	.3153	4987	3.1716	5013	30
40	.3035	4821	.9528	9790	.3185	5031	3.1397	4969	20
50	.3062	4861	.9620	9786	.3217	5075	3.1084	4925	10
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE XVI—*continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
10	.3118	4939	.9502	9778	.3281	5161	3.0475	4839	50
20	.3145	4977	.9492	9774	.3314	5203	3.0178	4797	40
30	.3173	5015	.9483	9770	.3346	5245	2.9887	4755	30
40	.3201	5052	.9474	9765	.3378	5287	2.9600	4713	20
50	.3228	5090	.9465	9761	.3411	5329	2.9319	4671	10
19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.9042	0.4630	71° 00′
10	.3283	5163	.9446	9752	.3476	5411	2.8770	4589	50
20	.3311	5199	.9436	9748	.3508	5451	2.8502	4549	40
30	.3338	5235	.9426	9743	.3541	5491	2.8239	4509	30
40	.3365	5270	.9417	9739	.3574	5531	2.7980	4469	20
50	.3393	5306	.9407	9734	.3607	5571	2.7725	4429	10
20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.7475	0.4389	70° 00′
10	.3448	5375	.9387	9725	.3673	5650	2.7228	4350	50
20	.3475	5409	.9377	9721	.3706	5689	2.6985	4311	40
30	.3502	5443	.9367	9716	.3739	5727	2.6746	4273	30
40	.3529	5477	.9356	9711	.3772	5766	2.6511	4234	20
50	.3557	5510	.9346	9706	.3805	5804	2.6279	4196	10
21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	2.6051	0.4158	69° 00′
10	.3611	5576	.9325	9697	.3872	5879	2.5826	4121	50
20	.3638	5609	.9315	9692	.3906	5917	2.5605	4083	40
30	.3665	5641	.9304	9687	.3939	5954	2.5386	4046	30
40	.3692	5673	.9293	9682	.3973	5991	2.5172	4009	20
50	.3719	5704	.9283	9677	.4006	6028	2.4960	3972	10
22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.4751	0.3936	68° 00′
10	.3773	5767	.9261	9667	.4074	6100	2.4545	3900	50
20	.3800	5798	.9250	9661	.4108	6136	2.4342	3864	40
30	.3827	5828	.9239	9656	.4142	6172	2.4142	3828	30
40	.3854	5859	.9228	9651	.4176	6208	2.3945	3792	20
50	.3881	5889	.9216	9646	.4210	6243	2.3750	3757	10
23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.3559	0.3721	67° 00′
10	.3934	5948	.9194	9635	.4279	6314	2.3369	3686	50
20	.3961	5978	.9182	9629	.4314	6348	2.3183	3652	40
30	.3987	6007	.9171	9624	.4348	6383	2.2998	3617	30
40	.4014	6036	.9159	9618	.4383	6417	2.2817	3583	20
50	.4041	6065	.9147	9613	.4417	6452	2.2637	3548	10
24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.2460	0.3514	66° 00′
10	.4094	6121	.9124	9602	.4487	6520	2.2286	3480	50
20	.4120	6149	.9112	9596	.4522	6553	2.2113	3447	40
30	.4147	6177	.9100	9590	.4557	6587	2.1943	3413	30
40	.4173	6205	.9088	9584	.4592	6620	2.1775	3380	20
50	.4200	6232	.9075	9579	.4628	6654	2.1609	3346	10
25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.1445	0.3313	65° 00′
10	.4253	6286	.9051	9567	.4699	6720	2.1283	3280	50
20	.4279	6313	.9038	9561	.4734	6752	2.1123	3248	40
30	.4305	6340	.9026	9555	.4770	6785	2.0965	3215	30
40	.4331	6366	.9013	9549	.4806	6817	2.0809	3183	20
50	.4358	6392	.9001	9543	.4841	6850	2.0655	3150	10
26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.0503	0.3118	64° 00′
10	.4410	6444	.8975	9530	.4913	6914	2.0353	3086	50
20	.4436	6470	.8962	9524	.4950	6946	2.0204	3054	40
30	.4462	6495	.8949	9518	.4986	6977	2.0057	3023	30′
40	.4488	6521	.8936	9512	.5022	7009	1.9912	2991	20
50	.4514	6546	.8923	9505	.5059	7040	1.9768	2960	10
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE XVI—*continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
10	.4566	6595	.8897	9492	.5132	7103	1.9486	2897	50
20	.4592	6620	.8884	9486	.5169	7134	1.9347	2866	40
30	.4617	6644	.8870	9479	.5206	7165	1.9210	2835	30
40	.4643	6668	.8857	9473	.5243	7196	1.9074	2804	20
50	.4669	6692	.8843	9466	.5280	7226	1.8940	2774	10
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.8807	0.2743	62° 00′
10	.4720	6740	.8816	9453	.5354	7287	1.8676	2713	50
20	.4746	6763	.8802	9446	.5392	7317	1.8546	2683	40
30	.4772	6787	.8788	9439	.5430	7348	1.8418	2652	30
40	.4797	6810	.8774	9432	.5467	7378	1.8291	2622	20
50	.4823	6833	.8760	9425	.5505	7408	1.8165	2592	10
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.8040	0.2562	61° 00′
10	.4874	6878	.8732	9411	.5581	7467	1.7917	2533	50
20	.4899	6901	.8718	9404	.5619	7497	1.7796	2503	40
30	.4924	6923	.8704	9397	.5658	7526	1.7675	2474	30
40	.4950	6946	.8689	9390	.5696	7556	1.7556	2444	20
50	.4975	6968	.8675	9383	.5735	7585	1.7437	2415	10
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.7321	0.2386	60° 00′
10	.5025	7012	.8646	9368	.5812	7644	1.7205	2356	50
20	.5050	7033	.8631	9361	.5851	7673	1.7090	2327	40
30	.5075	7055	.8616	9353	.5890	7701	1.6977	2299	30
40	.5100	7076	.8601	9346	.5930	7730	1.6864	2270	20
50	.5125	7097	.8587	9338	.5969	7759	1.6753	2241	10
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.6643	0.2212	59° 00′
10	.5175	7139	.8557	9323	.6048	7816	1.6534	2184	50
20	.5200	7160	.8542	9315	.6088	7845	1.6426	2155	40
30	.5225	7181	.8526	9308	.6128	7873	1.6319	2127	30
40	.5250	7201	.8511	9300	.6168	7902	1.6212	2098	20
50	.5275	7222	.8496	9292	.6208	7930	1.6107	2070	10
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.6003	0.2042	58° 00′
10	.5324	7262	.8465	9276	.6289	7986	1.5900	2014	50
20	.5348	7282	.8450	9268	.6330	8014	1.5798	1986	40
30	.5373	7302	.8434	9260	.6371	8042	1.5697	1958	30
40	.5398	7322	.8418	9252	.6412	8070	1.5597	1930	20
50	.5422	7342	.8403	9244	.6453	8097	1.5497	1903	10
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.5399	0.1875	57° 00′
10	.5471	7380	.8371	9228	6536	8153	1.5301	1847	50
20	.5495	7400	.8355	9219	.6577	8180	1.5204	1820	40
30	.5519	7419	.8339	9211	.6619	8208	1.5108	1792	30
40	.5544	7438	.8323	9203	.6661	8235	1.5013	1765	20
50	.5568	7457	.8307	9194	.6703	8263	1.4919	1737	10
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.4826	0.1710	56° 00′
10	.5616	7494	.8274	9177	.6787	8317	1.4733	1683	50
20	.5640	7513	.8258	9169	.6830	8344	1.4641	1656	40
30	.5664	7531	.8241	9160	.6873	8371	1.4550	1629	30
40	.5688	7550	.8225	9151	.6916	8398	1.4460	1602	20
50	.5712	7568	.8208	9142	.6959	8425	1.4370	1575	10
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.4281	0.1548	55° 00′
10	.5760	7604	.8175	9125	.7046	8479	1.4193	1521	50
20	.5783	7622	.8158	9116	.7089	8506	1.4106	1494	40
30	.5807	7640	.8141	9107	.7133	8533	1.4019	1467	30
40	.5831	7657	.8124	9098	.7177	8559	1.3934	1441	20
50	.5854	7675	.8107	9089	.7221	8586	1.3848	1414	10
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE XVI—*concluded*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
10	.5901	7710	.8073	9070	.7310	8639	1.3680	1361	50
20	.5925	7727	.8056	9061	.7355	8666	1.3597	1334	40
30	.5948	7744	.8039	9052	.7400	8692	1.3514	1308	30
40	.5972	7761	.8021	9042	.7445	8718	1.3432	1282	20
50	.5995	7778	.8004	9033	.7490	8745	1.3351	1255	10
37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.3270	0.1229	53° 00′
10	.6041	7811	.7969	9014	.7581	8797	1.3190	1203	50
20	.6065	7828	.7951	9004	.7627	8824	1.3111	1176	40
30	.6088	7844	.7934	8995	.7673	8850	1.3032	1150	30
40	.6111	7861	.7916	8985	.7720	8876	1.2954	1124	20
50	.6134	7877	.7898	8975	.7766	8902	1.2876	1098	10
38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.2799	0.1072	52° 00′
10	.6180	7910	.7862	8955	.7860	8954	1.2723	1046	50
20	.6202	7926	.7844	8945	.7907	8980	1.2647	1020	40
30	.6225	7941	.7826	8935	.7954	9006	1.2572	0994	30
40	.6248	7957	.7808	8925	.8002	9032	1.2497	0968	20
50	.6271	7973	.7790	8915	.8050	9058	1.2423	0942	10
39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.2349	0.0916	51° 00′
10	.6316	8004	.7753	8895	.8146	9110	1.2276	0890	50
20	.6338	8020	.7735	8884	.8195	9135	1.2203	0865	40
30	.6361	8035	.7716	8874	.8243	9161	1.2131	0839	30
40	.6383	8050	.7698	8834	.8292	9187	1.2059	0813	20
50	.6406	8066	.7679	8853	.8342	9212	1.1988	0788	10
40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.1918	0.0762	50° 00′
10	.6450	8096	.7642	8832	.8441	9264	1.1847	0736	50
20	.6472	8111	.7623	8821	.8491	9289	1.1778	0711	40
30	.6494	8125	.7604	8810	.8541	9315	1.1708	0685	30
40	.6517	8140	.7585	8800	.8591	9341	1.1640	0659	20
50	.6539	8155	.7566	8789	.8642	9366	1.1571	0634	10
41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.1504	0.0608	49° 00′
10	.6583	8184	.7528	8767	.8744	9417	1.1436	0583	50
20	.6604	8198	.7509	8756	.8796	9443	1.1369	0557	40
30	.6626	8213	.7490	8745	.8847	9468	1.1303	0532	30
40	.6648	8227	.7470	8733	.8899	9494	1.1237	0506	20
50	.6670	8241	.7451	8722	.8952	9519	1.1171	0481	10
42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.1106	0.0456	48° 00′
10	.6713	8269	.7412	8699	.9057	9570	1.1041	0430	50
20	.6734	8283	.7392	8688	.9110	9595	1.0977	0405	40
30	.6756	8297	.7373	8676	.9163	9621	1.0913	0379	30
40	.6777	8311	.7353	8665	.9217	9646	1.0850	0354	20
50	.6799	8324	.7333	8653	.9271	9671	1.0786	0329	10
43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.0724	0.0303	47° 00′
10	.6841	8351	.7294	8629	.9380	9722	1.0661	0278	50
20	.6862	8365	.7274	8618	.9435	9747	1.0599	0253	40
30	.6884	8378	.7254	8606	.9490	9772	1.0538	0228	30
40	.6905	8391	.7234	8594	.9545	9798	1.0477	0202	20
50	.6926	8405	.7214	8582	.9601	9823	1.0416	0177	10
44° 00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	1.0355	0.0152	46° 00′
10	.6967	8431	.7173	8557	.9713	9874	1.0295	0126	50
20	.6988	8444	.7153	8545	.9770	9899	1.0235	0101	40
30	.7009	8457	.7133	8532	.9827	9924	1.0176	0076	30
40	.7030	8469	.7112	8520	.9884	9949	1.0117	0051	20
50	.7050	8482	.7092	8507	.9942	9975	1.0058	0025	10
45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	45° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE XVII.—WORKING STRESSES IN POUNDS PER SQUARE INCH

Recommended by Forest Products Laboratory United States Forest Service

Exposure	Kind of timber	Bending		Compression		Horizontal shear	Intermediate values of *K*		Modulus of elasticity *E*
		b = 4″ or less	b = 5″ or more	Parallel	Normal		Select	Common	
Continuously Dry	Douglas fir, Coast Region, select	1,600	1,600	1,175	345	90	23.7	27.3	1,600,000
	Douglas fir, Coast Region, dense select	1,750	1,750	1,290	380	105	22.6	24.9	1,600,000
	Douglas fir, Rocky Mt. Region, select	1,100	1,100	800	275	85	24.8	27.8	1,200,000
	Hemlock, West Coast, select	1,300	1,300	900	300	75	25.3	28.3	1,400,000
	Hemlock, Eastern, select	1,100	1,100	700	300	70	25.4	....	1,100,000
	Pine, Southern, select	1,600	1,600	1,175	345	110	23.7	27.3	1,600,000
	Pine, Southern, dense select	1,750	1,750	1,290	380	128	22.6	24.9	1,600,000
Occasionally wet, quickly dried	Douglas fir, Coast Region, select	1,240	1,385	1,065	240	90	24.9	28.6	1,600,000
	Douglas fir, Coast Region, dense select	1,370	1,515	1,165	265	105	23.8	26.1	1,600,000
	Douglas fir, Rocky Mt. Region, select	800	900	800	225	85	24.8	27.8	1,200,000
	Hemlock, West Coast, select	980	1,100	900	225	75	25.3	28.3	1,400,000
	Hemlock, Eastern, select	800	.....	.....	225	70			
	Pine, Southern, select	1,240	1,385	1,065	240	110	24.9	28.6	1,600,000
	Pine, Southern, dense select	1,370	1,515	1,165	265	128	23.8	26.1	1,600,000
Usually damp or wet	Douglas fir, Coast Region, select	950	1,065	905	215	90	27.0	31.1	1,600,000
	Douglas fir, Coast Region, dense select	1,050	1,165	990	235	105	25.8	28.3	1,600,000
	Douglas fir, Rocky Mt. Region, select	620	700	700	200	85	26.5	29.7	1,200,000
	Hemlock, West Coast, select	800	900	800	200	75	26.8	30.0	1,400,000
	Hemlock, Eastern, select	710	.....	.....	200	70			
	Pine, Southern, select	950	1,065	905	215	110	27.0	31.1	1,600,000
	Pine, Southern, dense select	1,050	1,165	990	235	128	25.8	28.3	1,600,000

Working Stresses for Common Grades may be taken equal to 80 per cent of the values above.

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ANSWERS

CHAPTER I

1. 4,000 lb./sq.in.
2. 6,360 lb./sq. in.
3. 204 lb./sq. in.
4. 833 lb./sq. in.
5. 22.2 sq. in.
6. 2,730 lb./sq. in.
7. 363 lb./sq. in.
8. 100 lb./sq. in.
9. 18,180 lb./sq. in.
10. 0.28 in.
11. 48,000 lb./sq. in.
12. 11,620 lb./sq. in.
13. 21.800 lb./sq. in., 5,090 lb./sq. in.
14. 28,100 lb./sq. in., 6,360 lb./sq. in.
15. 5,380 lb./sq. in.
16. 10,560 lb./sq. in.
17. 96 ft.
18. 120 ft.
19. 110.7 ft.
20. 120 sq. in., 13.68 sq. ft.
21. 2.24 in., 0.437 in.
22. 2,115 lb./sq. in.
23. 140 sq. in., 14.48 sq. in.
24. 40 sq. in., 801 sq. in., 52.78 sq. ft.
25. 0.66 in.
26. 62,830 lb.
28. 50,400 lb.
30. 0.44 in.
31. 220 lb./sq. in.
33. $\begin{cases} d = \frac{3}{4} \text{ in.} \\ h = 0.12 \text{ in.} \end{cases}$
34. 30.83 ft.
35. 12,000 lb./sq. in.

CHAPTER II

37. $l = 2.67$ in.
38. Yes.
39. $l = 2$ in.
40. 271 lb./sq. in.
41. 10.
42. 7,130 lb.
43. 4.
44. 5.
45. 15,820 lb.
46. 6.
47. $\begin{cases} S_s = 10{,}000 \text{ lb./sq. in.} \\ S_c = 18{,}300 \text{ lb./sq. in.} \\ S_t = 10{,}250 \text{ lb./sq. in.} \end{cases}$
48. $\begin{cases} 5{,}010 \text{ lb./sq. in.} \\ d = 1.38 \text{ in.} \\ p = 6.13 \text{ in.} \\ b = 14 \text{ ft.} \end{cases}$
49. 15, 21.
51. 4, 3.
53. 0.96 in.
54. 69,200 lb. 88,100. 0.01175 in.
56. Use $\frac{5}{32}$ for 3-ft. tank.
57. 3, 2, 3.
58. Yes. 78,300 lb.

62. $\begin{cases} d = 1 \text{ in.} \\ t = 0.4 \text{ in.} \\ w = 3.56 \text{ in.} \end{cases}$
64. Yes.
65. Yes.
67. 16,300 lb.
68. 2.5 in.
69. 15,700 lb.
71. $\begin{cases} S_t = 9{,}280 \text{ lb./sq. in.} \\ S_s = 6{,}840 \text{ lb./sq. in.} \\ s_c = 14{,}700 \text{ lb./sq. in.} \end{cases}$

CHAPTER III

72. 0.076 in.,
15,250 lb./sq. in.
73. .00045 in.,
13,050 lb./sq. in.
74. 54,000 lb.
75. 0.00097.
77. 4.
78. 60.7 in.
79. 0.0088.
81. No. 0.75 in. 0.004.
82. 113.6 in.
83. 0.0109.
84. 5,610. 17,160.
85. 0.027.
86. 590, 8,860. 4.75 ft.
87. 0.129 in. 9,760.
89. 212, 3,150. 945.
91. 0.098 in. for 50 ft. 260.
92. $r = 1.17$ in.
93. 24, 760,000.
95. 0.0295.
97. 100.0345 ft.

CHAPTER IV

98. $\begin{cases} AC = 10{,}390. \\ BC = 6{,}000. \end{cases}$
99. $\begin{cases} BC = 10. \\ BA = 6.16. \end{cases}$
100. 184.7 lb., 92.4 lb.
101. 23.3 lb., 30°58′.
102. 106.9 lb., 24°42′.
104. 83.56 lb.
105. 3.2, 2.5.
106. 2,795. 1,863.
108. 662, 280.
109. 1,050.
110. 83.3.
111. 203. 35.3.
112. 4.83.
113. 1.37, 1.07.
114. 3,200, 2,940.
116. 288, 577.
117. $200 + 0.1W$.
119. 100, 173.
121. 151.5.
123. 1,610, 6,220.
124. 1,572.
126. $\begin{cases} R_B = 57.7. \\ R_A = 208.1. \end{cases}$
127. 833 lb.

CHAPTER V

128. 350, 250.
129. 77.5, 47.5.
130. 8,500, 13,500.
131. 2,280, 2,160.
132. 10,130, 3,070.
133. 5,520, 7,080.
135. 1034.2.
136. 4,480.
137. 45.4. 90.8.
140. 60 lb.
141. 10.
142. 100.
143. 20 ft.
147. 180 lb. in.

CHAPTER VI

151. 18 lb.
152. 38.
153. 12 lb. 53.
154. 0.2.
155. 0.118.
156. 11°.
157. No. 31.8. 103.2.
158. 57.2, 42.9, 14.3.
159. 19° 16′. 6° 25′.
160. 35 in.
161. 810. 558.
165. 37.5.
166. 0.364.
167. 26.6 lb.
168. No.
171. 1,820 lb.

CHAPTER VII

173. 8.25. 16.5.
174. 6 by 6 in.
176. 2.25, 4.5.
177. $d = 1.1$. 10,600.

CHAPTER VIII

179. 23, 260. 120.
180. 8.66, 17.32, 45.0.
181. 2,550. 3,400.
184. 1,000, 1,333.
186. $\left\{\begin{array}{l} H = 685. \\ V = 390. \end{array}\right.$
189. 1,158. 14,140, 10,000, 7,500.
190. 1,440.
191. $\left\{\begin{array}{l} A_H = C_H = 190. \\ A_V = 289. \\ C_V = 211. \end{array}\right.$
193. $A = B = C = 28{,}000$. 8,480.
194. 22.

CHAPTER IX

No answers.

CHAPTER X

211. 1.16.
213. 12.7.
216. 5.24.
217. 5.28.
220. 4.67, 7.33.
221. 100.
225. 1.68, 4.26.
227. 6.05, 3.67.
230. 2.08,
232. 3.05, 5.24.

CHAPTER XI

233. 8.22, 7.85.
235. 1,866.5.

CHAPTER XII

240. 3,910.
241. 1,413.
242. 10.8 ft.
244. 10 in., 25.4 lb.
245. 12 in., 31.8.
247. 5.6 in.
250. $\left\{\begin{array}{l} y = 3.22. \\ I = 155.0. \end{array}\right.$ 636, 946.
251. 9,630.

253. 2.26.
254. $I = 1{,}620$.
$W = 18{,}515$.
255. 15 in.
256. 75 lb.
258. 9 in., 25 lb., 15 in., 60 lb.
260. 156 lb./sq. ft.
266. 13.35.
272. 1,039.
274. 24,600.
277. 6 in., 10.5 lb.
280. 7.4 in.
282. 6 in., 12.5 lb.

CHAPTER XIII

283. 31.0. 23.4. 1,125.
286. 87.0. 50.0.
290. 20,530. 754.0.
293. 11.54.
294. 0.5 in.
295. 10.46 in.
303. 3.33, 6.66.
305. 31.0. 23.4.
307. 5,850.

CHAPTER XIV

308. 1,418, 79, 0.528.
311. 25:9.
312. 6.
314. 18,560, 2,435, 0.616.
317. 4,280, 340, 0.0043.
318. 18 in. 54.7 lb.

CHAPTER XV

321. 4 by 4.
324. 715.
325. 9.
326. 9,600.
327. 2,790.
331. 625, 875.
333. 4,090, 11,150.
336. 2.91.
338. 0.875.
343. 5.51 ft. from toe.
2.53, 0.089.

CHAPTER XVI

347. 5 in., 10 lb., 5,560 ft./lb., 3,700 ft./lb.
348. 6 in., 12.5 lb.
349. 600 lb./ft.
350. 694, 96.
351. 1,780, 1,430, 115.
353. 3,330.
356. 10 by 15 in.
357. 4 in.

CHAPTER XVII

365. 5.75 in.
367. 100,100.
370. 3.38.
373. 18.5.
378. 10,340. 12.8.
386. 614,000.

CHAPTER XVIII

389. 124.0.
390. 3.35.
391. 208.5.
395. 1:8:27.
396. 1,745, 1,070.
401. 2,480, 6,220.
402. 85.5.
404. 0.25 by 0.25 in.
405. 22,000 ft.-lb.
409. 0.44°. 1,930.

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